Fibonacci Number of the Tadpole Graph

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Recommended Citation
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Fibonacci number of the tadpole graph

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Abstract

In 1982, Prodinger and Tichy defined the Fibonacci number of a graph \( G \) to be the number of independent sets of the graph \( G \). They did so since the Fibonacci number of the path graph \( P_n \) is the Fibonacci number \( F_{n+2} \) and the Fibonacci number of the cycle graph \( C_n \) is the Lucas number \( L_n \). The tadpole graph \( T_{n,k} \) is the graph created by concatenating \( C_n \) and \( P_k \) with an edge from any vertex of \( C_n \) to a pendant of \( P_k \) for integers \( n = 3 \) and \( k = 0 \). This paper establishes formulae and identities for the Fibonacci number of the tadpole graph via algebraic and combinatorial methods.

Keywords: independent sets; Fibonacci sequence; cycles; paths
Mathematics Subject Classification : 05C69

1. Introduction

Given a graph \( G = (V, E) \), a set \( S \subseteq V \) is an independent set of vertices if no two vertices in \( S \) are adjacent. In our illustrations, we indicate membership in an independent set \( S \) by shading the vertices in \( S \). Let the set of all independent sets of a graph \( G \) be denoted by \( I(G) \) and let \( i(G) = |I(G)| \). Note that \( \emptyset \in I(G) \). The path graph, \( P_n \), consists of the vertex set \( V = \{1, 2, \ldots, n\} \) and the edge set \( E = \{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}\} \). The cycle graph, \( C_n \), is the path graph, \( P_n \), with the additional edge \( \{1, n\} \).

Table 1 shows initial Fibonacci and Lucas numbers. In 1982, Prodinger and Tichy defined the Fibonacci number of a graph \( G \), \( i(G) \), to be the number of independent sets (including the empty
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Figure 1. Independent sets of $P_1$; $P_2$; $P_3$; and $P_4$.

Figure 2. Independent sets of $C_3$ and $C_4$.

set) of the graph $G$ [5]. They did so because the Fibonacci number of the path graph $P_n$ is the Fibonacci number $F_{n+2}$, and the Fibonacci number of the cycle graph $C_n$ is the Lucas number $L_n$.

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Table 1: Initial values of the Fibonacci and Lucas sequences

In [1], the authors of this paper use these graphs to combinatorially derive identities relating Fibonacci and Lucas numbers.

**Example 1.** $L_n = F_{n-1} + F_{n+1}$ for positive integers $n \geq 3$.

**Proof.** On the one hand we know that $i(C_n) = L_n$. On the other hand, vertex 1 is either a member of the independent set or it is not. If not, then any independent set from $P_{n-1}$, formed by vertices 2 through $n$, can be selected in $i(P_{n-1})$ ways. If in the set, then the remaining members can be selected in $i(P_{n-3})$ ways from the path formed by vertices 3 through $n-1$, since vertices 2 and $n$ can not be selected. Hence, $L_n = i(C_n) = i(P_{n-3}) + i(P_{n-1}) = F_{n-1} + F_{n+1}$.

The Fibonacci sequence and the Lucas sequence are famous examples of the more general form called the Gibonacci sequence [3]. For integers $G_0 = a$ and $G_1 = b$, the Gibonacci sequence is defined recursively as $G_n = G_{n-1} + G_{n-2}$ for positive integers $n \geq 2$. Do other graphs exist whose Fibonacci numbers form a Gibonacci sequence?
The tadpole graph, $T_{n,k}$, is the graph created by concatenating $C_n$ and $P_k$ with an edge from any vertex of $C_n$ to a pendent of $P_k$ for integers $n \geq 3$ and $k \geq 0$. For ease of reference we label the vertices of the cycle $c_1, \ldots, c_n$, the vertices of the path $p_1, \ldots, p_k$ where $c_1$ is adjacent to $p_1$.

![Figure 3. The Tadpole Graph $T_{n,k}$.](image-url)

**Example 2. Independent sets on $T_{3,2}$**

![Figure 4. $T_{3,2}$.](image-url)

$I(T_{3,2}) = \{\emptyset, \{c_1\}, \{c_2\}, \{c_3\}, \{p_1\}, \{p_2\}, \{c_1, p_2\}, \{c_2, p_1\}, \{c_2, p_2\}, \{c_3, p_1\}, \{c_3, p_2\}\}$

**Theorem 1.1.** $i(T_{n,k}) = i(T_{n,k-1}) + i(T_{n,k-2})$.

**Proof.** We show that $I(T_{n,k}) = I(T_{n,k-1}) \cup I(T_{n,k-2})$ where $I(T_{n,k-1}) \cap I(T_{n,k-2}) = \emptyset$. Partition $I(T_{n,k})$ into two disjoint subsets: sets where $p_k$ is shaded and sets where $p_k$ is not shaded. For every independent set in $I(T_{n,k-2})$, add an unshaded vertex $p_{k-1}$ followed by a shaded vertex $p_k$ to the end of the path graph. For every independent set in $I(T_{n,k-1})$, add an unshaded vertex $p_k$ to the end of the path graph. Therefore, $i(T_{n,k}) = i(T_{n,k-1}) + i(T_{n,k-2})$. \qed

**Theorem 1.2.** $i(T_{n,k}) = i(T_{n-1,k}) + i(T_{n-2,k})$.

**Proof.** Again, we show that $I(T_{n,k}) = I(T_{n-1,k}) \cup I(T_{n-2,k})$ where $I(T_{n-1,k}) \cap I(T_{n-2,k}) = \emptyset$. Label any three consecutive vertices of $T_{n,k}$ of degree two from the cycle as $n-1$, $n$, and 1. For every independent set in $I(T_{n-2,k})$, if vertex 1 is shaded (then vertex $n-2$ is not shaded), insert a shaded vertex $n-1$ and an unshaded vertex $n$, thus creating all independent sets of $T_{n,k}$ that include both 1 and $n-1$. If vertex 1 is not shaded, then insert a shaded vertex $n$ and unshaded vertex $n-1$ creating all independent sets where vertex $n$ is shaded. For every independent set in $I(T_{n-1,k})$, insert an unshaded vertex $n$ which finally creates all independent sets where either 1 or $n-1$ is shaded or none of $n-1$, $n$, and 1 are shaded. Therefore, $i(T_{n,k}) = i(T_{n-1,k}) + i(T_{n-2,k})$. \qed
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It is immediate that \( i(T_{n,0}) = L_n \) since \( T_{n,0} \cong C_n \). Computing \( i(T_{3,2}) = 11 \) and \( i(T_{4,1}) = 12 \) allows us to effortlessly fill in the following table since by Theorems 1.1 and 1.2, every row and column forms a Gibonacci sequence.

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Table 2: Fibonacci Numbers for the Tadpole Graph, \( T_{n,k} \)

It is easy to directly compute the Fibonacci number of any Tadpole graph.

**Theorem 1.3.** \( i(T_{n,k}) = L_{n+k} + F_{n-3}F_k \).

**Proof.** We proceed with two base cases and strong induction on \( k \). Suppose \( k = 0 \). Then \( i(T_{n,0}) = i(C_n) = L_n + F_{n-3}F_0 = L_n \). For \( k = 1 \), combinatorially, \( i(T_{n,1}) = i(C_n) + i(P_{n-1}) = L_n + F_{n+1} \). Now algebraically,

\[
L_n + F_{n+1} = L_{n+1} - L_{n-1} + F_n + F_{n-1}
= L_{n+1} - F_{n-2} + F_{n-1}
= L_{n+1} + F_{n-3}F_1.
\]

Finally, for general \( k \geq 2 \),

\[
i(T_{n,k+1}) = i(T_{n,k}) + i(T_{n,k-1})
= L_{n+k} + F_{n-3}F_k + L_{n+k-1} + F_{n-3}F_{k-1}
= L_{n+k+1} + F_{n-3}F_{k+1}.
\]

**Theorem 1.4.** For \( n \geq 3 \) and \( k \geq 0 \),

1. \( L_{n+k} = F_{n-1}F_{k+1} + F_{n+1}F_{k+2} - F_{n-3}F_k \);
2. \( L_{n+k} = F_{n+1}F_k + L_nF_{k+1} - F_{n-3}F_k \);
3. \( L_{n+k} = F_{n-1}F_{k+2} + F_{n+k+1} - F_{n-3}F_k \).
Proof. For 1, we know that there are $L_{n+k} + F_{n-3}F_k$ independent sets on the tadpole graph $T_{n,k}$. Now we partition $I(T_{n,k})$ into two disjoint sets: sets that contain $c_1$ and sets that do not. If $c_1$ is included in the independent set then $c_2, c_3$ and $p_1$ are not. Hence, there are $i(P_{n-3})i(P_{k-1}) = F_{n-1}F_{k+1}$ such sets. If $c_1$ is not included in the independent set then there are $i(P_{n-1})i(P_k) = F_{n+1}F_{k+2}$ such sets. So, $L_{n+k} + F_{n-3}F_k = F_{n-1}F_{k+1} + F_{n+1}F_{k+2}$ and the result follows.

For 2, we partition $I(T_{n,k})$ into two disjoint sets: sets that contain $p_1$ and sets that do not. If $p_1$ is included in the independent set then $c_1$ and $p_2$ are not. Hence, there are $i(P_{n-1})i(P_{k-2}) = F_{n+1}F_k$ such sets. If $p_1$ is not included in the independent set then there are $i(C_n)i(P_{k-1}) = L_nF_{k+1}$ such sets. So $L_{n+k} + F_{n-3}F_k = F_{n+1}F_k + L_nF_{k+1}$ and the result follows.

For 3, we partition $I(T_{n,k})$ into two disjoint sets: sets that contain $c_n$ and sets that do not. If $c_n$ is included in the independent set then $c_1$ and $c_{n-1}$ are not. Hence, there are $i(P_{n-3})i(P_k) = F_{n-1}F_{k+2}$ such sets. If $c_n$ is not included in the independent set then there are $i(P_{n-1+k}) = F_{n+k+1}$ such sets. So, $L_{n+k} + F_{n-3}F_k = F_{n-1}F_{k+2} + F_{n+k+1}$ and the result follows.

\[\square\]

2. Tadpole Triangle

We turn Table 2 into a triangular array where the $(n, k)$ entry for $n \geq 3$ and $k \geq 0$ will be denoted $t_{n,k}$. Row $n$ will represent the class of tadpole graphs with a total of $n$ vertices. As the value of $k$ increases by 1 through each row of the triangle, the cycle subgraph shrinks by one vertex and the length of the path subgraph increases by one. Thus, $t_{n,k}$ represents the number of independent sets on the Tadpole graph with $n$ vertices with a path of length $k$ (and thus, a cycle of length $n - k$). Hence, $t_{n,k} = i(T_{n-k,k})$. By Theorem 1.3, $t_{n,k} = L_n + F_{n-k-3}F_k$.

\[
\begin{array}{cccc}
4 & 7 & 7 \\
11 & 12 & 11 \\
18 & 19 & 19 & 18 \\
29 & 31 & 30 & 31 & 29 \\
47 & 50 & 49 & 49 & 50 & 47 \\
76 & 81 & 79 & 80 & 79 & 81 & 76 \\
\end{array}
\]

Table 3: The Triangular Array of Fibonacci Numbers of the Tadpole Graph

As noted before, $t_{n,0} = i(T_{n,0}) = L_n$. Casual observation seems to indicate the rows the tadpole triangle are symmetric.

**Theorem 2.1.** $t_{n,k} = t_{n,n-k-3}$

**Proof.** Theorem 1.3 provides a quick, algebraic proof of the symmetry of rows since $t_{n,n-k-3} = i(T_{k+3,n-k-3}) = L_n + F_kF_{n-k-3} = i(T_{n-k,k}) = t_{n,k}$. \[\square\]

**Proof.** For a combinatorial proof of the symmetry in rows, consider $c_2$ in both $T_{k+3,n-k-3}$ and $T_{n-k,k}$. As before, we partition the tadpole graphs into two disjoint sets: those that contain $c_2$ and those that do not. Both tadpole graphs contain $n$ vertices. Thus, the number of independent sets that do not contain $c_2$ in each tadpole graph is the number of independent sets on the path with
Proof. Algebraically,

\[ t_{n,k+1} - t_{n,k} = (-1)^k F_{n-2k-4} \text{ for } 0 \leq k \leq \left\lfloor \frac{n-3}{2} \right\rfloor. \]

**Theorem 2.2.** \( t_{n,k+1} - t_{n,k} = (-1)^k F_{n-2k-4} \text{ for } 0 \leq k \leq \left\lfloor \frac{n-3}{2} \right\rfloor. \)

*Proof.* Combinatorially we proceed by initially considering the mapping

\[
\Psi (S) = \begin{cases} 
S \text{ for } \{c_2, c_{n-k}\} \not\subseteq S \\
(S \setminus \{c_2\}) \cup \{c_1\} \text{ for } \{c_2, c_{n-k}\} \subseteq S
\end{cases}
\]

from \( I(T_{n-k,k}) \) to \( I(T_{n-k-1,k+1}) \) as illustrated in Figure 5. The identity mapping pairs together most independent sets but encounters obvious problems since independent sets in \( I(T_{n-k,k}) \) that contain both \( c_2 \) and \( c_{n-k} \) do not map to \( I(T_{n-k-1,k+1}) \), and independent sets in \( I(T_{n-k-1,k+1}) \) that contain both \( c_1 \) and \( c_{n-k} \) have no pre-image in \( I(T_{n-k,k}) \). If \( c_2 \) and \( c_{n-k} \) are both in the independent set, then remove \( c_2 \) from the independent set while including \( c_1 \) to create an independent set in \( I(T_{n-k-1,k+1}) \) to upgrade the identity mapping to \( \Psi (S) \). We now have two subtle issues which provide the value of \( t_{n,k+1} - t_{n,k} \). Independent sets in \( I(T_{n-k,k}) \) that contain the subset \( \{p_1, c_2, c_{n-k}\} \) have no image. There are \( i(P_{n-k-5})i(P_{k-2}) = F_{n-k-3}F_k \) such sets. Independent sets in \( I(T_{n-k-1,k+1}) \) that contain the subset \( \{c_1, c_2, c_{n-k}\} \) have no pre-image. There are \( i(P_{n-k-6})i(P_{k-1}) = F_{n-k-4}F_{k+1} \) such sets. Once again, \( t_{n,k+1} - t_{n,k} = F_{n-k-4}F_{k+1} - F_{n-k-3}F_k = (-1)^k F_{n-2k-4} \). \( \square \)

Figure 5. Mapping \( I(T_{n-k,k}) \) to \( I(T_{n-k-1,k+1}) \).
Theorem 2.3. \( \sum_{k=0}^{n-3} (-1)^k t_{n,k} = \begin{cases} 0 & \text{for even } n \\ 2F_n & \text{for odd } n. \end{cases} \)

Proof. For even \( n \) the result is trivial due to the symmetry of row values. For odd \( n \), we proceed by induction. Base cases abound from Table 3. Assume \( n \) is odd and \( \sum_{k=0}^{n-3} (-1)^k t_{n,k} = 2F_n \). Moving on to the next odd value we consider

\[
\sum_{k=0}^{n-1} (-1)^k t_{n+2,k} = \left( \sum_{k=0}^{n-3} (-1)^k t_{n+2,k} \right) + (-1)^{n-2} t_{n+2,n-2} + (-1)^{n-1} t_{n+2,n-1}
\]

\[
= \sum_{k=0}^{n-3} (-1)^k t_{n+2,k} - t_{n+2,n-2} + L_{n+2}
\]

\[
= \left( \sum_{k=0}^{n-3} (-1)^k [t_{n,k} + t_{n+1,k}] \right) - L_{n+2} + F_{n+2} - L_{n+2}
\]

\[
= \sum_{k=0}^{n-3} (-1)^k t_{n,k} + (-1)^k t_{n+1,k} - L_{n+2} - F_{n-2} + L_{n+2}
\]

\[
= \left( \sum_{k=0}^{n-3} (-1)^k t_{n,k} \right) + \left( \sum_{k=0}^{n-2} (-1)^k t_{n+1,k} \right) + t_{n+1,n-2} - L_{n+2} - F_{n-2} + L_{n+2}
\]

\[
= 2F_n + 0 + t_{n+1,n-2} - F_{n-2} = 2F_n + L_{n+1} - F_{n-2}
\]

\[
= 3F_n + F_{n+2} - F_{n-2} = 2F_n + F_{n-1} + F_{n+2}
\]

\[
= F_n + F_{n+1} + F_{n+2} = 2F_{n+2}
\]

Theorem 2.4. The ratio of consecutive row sums converges to the golden ratio \( \phi \).

Proof. The sum of row \( n \) can be written as

\[
\sum_{k=0}^{n-3} t_{n,k} = \sum_{k=0}^{n-3} (L_n + F_{n-k-3}F_k)
\]

\[
= \sum_{k=0}^{n-3} L_n + \sum_{k=0}^{n-3} F_{n-k-3}F_k
\]

\[
= (n - 2)L_n + \frac{(n - 3)L_{n-3} - F_{n-3}}{5}
\]
Now,

\[
\lim_{n \to \infty} \frac{(n + 1 - 2)L_{n+1} + \frac{(n+1-3)L_{n+1-3}-F_{n+1-3}}{5}}{(n - 2)L_n + \frac{(n-3)L_{n-3}-F_{n-3}}{5}} = \lim_{n \to \infty} \frac{(n - 1)L_{n+1} + \frac{(n-2)L_{n-2}-F_{n-2}}{5}}{(n - 2)L_n + \frac{(n-3)L_{n-3}-F_{n-3}}{5}} = \lim_{n \to \infty} \frac{nL_{n+1} + nL_{n-2} - F_{n-2}}{nL_n + nL_{n-3} - F_{n-3}} = \lim_{n \to \infty} \frac{L_{n+1} + L_{n-2}}{L_n + L_{n-3}} = \lim_{n \to \infty} \frac{L_n}{L_{n-1}} = \phi.
\]

A perfect matching (or 1-factor) in a graph \( G = (V, E) \) is a subset \( S \) of edges of \( E \) such that every vertex in \( V \) is incident to exactly one edge in \( S \). In [2], Gutman and Cyvin define the L-shaped graph, \( L_{p,q} \), to be the graph with \( p + q + 1 \) copies of \( C_4 \) as illustrated in Figure 6 by \( L_{2,1} \).
3. Future Work

At KSU, The Center for Excellence in Teaching and Learning’s URCA program and at BYU, The Center for Undergraduate Research in Mathematics both encourage the inclusion of undergraduate students in faculty research and dissemination of results at professional conferences.

Acknowledgement

The authors thank programs at Kennesaw State University and Brigham Young University for travel support when presenting this work at the Joint Mathematics Meetings in 2014 and the Forty-Fifth Southeastern International Conference on Combinatorics, Graph Theory, and Computing. At KSU, The Center for Excellence in Teaching and Learning’s URCA program and at BYU, The Center for Undergraduate Research in Mathematics both encourage the inclusion of undergraduate students in faculty research and dissemination of results at professional conferences.
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