Spring 5-6-2017

From Simplest Recursion to the Recursion of Generalizations of Cross Polytope Numbers

Yutong Yang
Kennesaw State University

From Simplest Recursion to the Recursion of Generalizations of Cross Polytope Numbers

Yutong Yang

Major: Mathematics, Kennesaw State University

May 6, 2017

Submitted to fulfill the partial requirement for the University Honors Program

Mathematics Advisor: Dr. Steven Edwards

Capstone Supervisor: Dr. Kami Anderson

Abstract

My research project involves investigations in the mathematical field of combinatorics. The research study will be based on the results of Professors Steven Edwards and William Griffiths, who recently found a new formula for the cross-polytope numbers. My topic will be focused on "Generalizations of cross-polytope numbers". It will include the proofs of the combinatorics results in Dr. Edwards and Dr. Griffiths' recently published paper. $E(n, m)$ and $O(n, m)$, the even terms and odd terms for Dr. Edward's original combinatorial expression, are two distinct combinatorial expressions that are in fact equal. But there is no obvious algebraic evidence to show that they are equal. There are induction proofs in the paper. But I wondered if there is a better way to explain that at the undergraduate level, so I proved it algebraically with combinatorial identities. $E_k(n, m)$ and $O_k(n, m)$, which are the generalized forms for $E(n, m)$ and $O(n, m)$, are in fact equal and share the same recurrence formula with $E(n, m)$ and $O(n, m)$. We can call those numbers from the table of E_k and O_k the generalizations of the cross-polytope numbers.

Contents

1 Introduction

Mathematical truths are given in theorems. Theoretical mathematicians advance mathematical knowledge by developing new principles and recognizing previously unknown relationships between existing principles of mathematics. Most importantly, theorems require proofs in order to be trusted.

Mathematicians are looking for newer better proofs of theorems. For examples, there are 367 proofs of the Pythagorean Theorem and has been republished by National Council of Teachers of Mathematics in 1968, and there are many different approaches and techniques to prove that there are infinitely many Primes. There are the ancient theorems that have been proved by people for thousands of years.

My main purpose of this research is to explore a different way to prove the recursion formulas that are given by Dr. Edwards and Dr. Griffiths, an algebraic proof based on the Pascal's Identity.

My research project involves investigations in the mathematical field of combinatorics. The research study will be based on the results of Professors Steven Edwards and William Griffiths, who recently found a new formula for the cross-polytope numbers. My topic will be focused on ”Generalizations of cross-polytope numbers”. It will include the proofs of the combinatorics results in Dr. Edwards and Dr. Griffiths' recent published paper.

$E(n, m)$ and $O(n, m)$, the even terms and odd terms for the Dr. Edward's original combinatorial expression, which are two distinct combinatorial expressions that are in fact equal. But there are no obvious evidence by algebra to show that they are equal. There are induction proofs in the paper. But I wonder if there is a better way to explain that at the undergraduate level, so I proved it algebraically with combinatorial identities. $E_k(n, m)$ and $O_k(n, m)$, which are the generalized forms for $E(n, m)$ and $O(n, m)$, are in fact equal and share the same recurrence formula with $E(n, m)$ and $O(n, m)$. We can call those numbers from the number table of E_k and O_k as the generalizations of the cross-polytope numbers.

2 Fundamentals

The following identities, definitions, theorems are the fundamentals to understand the proofs in this paper. It is important to know these following concepts in order to start studying combinatorics and number theory.

2.1 Fibonacci Identities

Definition The Fibonacci sequence is a series of numbers where a number is found by adding up the two numbers before it. Starting with 0 and 1, the sequence goes $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89$, ... and so forth. Written as a rule, the expression is defined by $f_0 = 0$, $f_1 = 1$, and for $n \geq 2$, $f_n = f_{n-1} + f_{n-2}$. (see [1]) As we go farther and farther to the right in this sequence, the ratio of a term to the one before it will get closer and closer to the **Golden Ratio**. In terms of mathematical expression: $\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \varphi$, where $\varphi = \frac{1+\sqrt{5}}{2}$.

The Fibonacci sequence can be extended indefinitely by applying the recursion relation. It may also extended as the index number n equals to negative values, by applying to the recursion relation. Extending negative n gives sequence ...$34, -21, 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13$... (see [2]). The reflection property can be summarized as $f_{-n} = (-1)^{n+1} f_n$

2.2 Binomial Coefficient

Definition The binomial coefficient $\binom{n}{k}$ is the number of ways of picking k unordered outcomes from n possibilities, also known as a combination or combinatorial number. The symbols ${}_nC_k$ and $\binom{n}{k}$ are used to denote a binomial coefficient, and are sometimes read as "n choose k." (see [1])

Examples

$$\binom{5}{1} = 5, \binom{4}{2} = 6, \binom{6}{4} = 15$$

If there are 4 students and I want to two students work together on a project, how many different groups that I have have? Well, we can solve this problem by binomial coefficient. Say the 4 students are $(1, 2, 3, 4)$, then I can group them in 6 different ways: $(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)$, which can be represented by $\binom{4}{2}$.

2.3 Combinatorial Interpretation of Binomial Coefficients

We use binomial coefficients to count. $\binom{n}{k}$ counts the ways to select a subgroup of k people from a total number of n people. By definition, we have $n \geq 0$, $\binom{n}{0} = 1$, and for $k < 0$, $\binom{n}{k} = 0$. We will define negative n in later section. The following identities can be found in [1].

Definition The value of the binomial coefficient for nonnegative n and k is given explicitly by (algebraic formula)

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

Exmaple

$$\binom{6}{2} = \frac{6!}{4!2!} = \frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1).(2 \cdot 1)} = 15$$

Identity 1 For $0 \leq k \leq n$,

$$\binom{n}{k} = \binom{n}{n-k}$$

Proof

$$\binom{n}{n-k} = \frac{n!}{(n-(n-k))!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Example

$$\binom{6}{2} = \binom{6}{4} = 15$$

Identity 2 For $0 \leq k \leq n$, (except $n = k = 0$)

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

2.4 Negative n

In previous section, we talked about the algebraic formula for $\binom{n}{k}$ when $n \geq 0$, how about $n < 0$?

It is actually defined as follows:

$$\binom{n}{k} = \frac{n(n-1)...(n-(k-1))}{k!} = (-1)^k \binom{k-n-1}{k}$$

Example

$$\binom{-6}{2} = \frac{(-6)(-7)}{2 \cdot 1} = 21$$

2.5 Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

for $1 \leq k \leq n$

We can use the identity to create a table, which is known as Pascal's triangle

$n = 0$:						1					
$n = 1$:					1		1				
$n = 2$:				1		2		1			
$n = 3$:			1		3		3		1		
$n = 4$:		1		4		6		4		1	
$n = 5$:	1		5		10		10		5		1

Algebraic proof

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!}$$

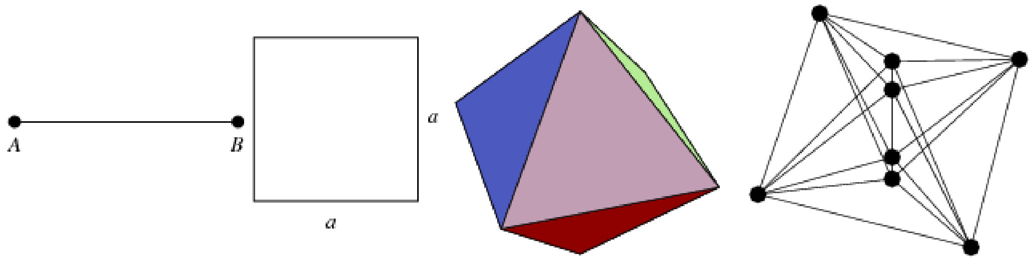
$$= (n-1)! \left[\frac{1}{k!(n-k-1)!} + \frac{1}{(k-1)!(n-k)!} \right]$$

$$= (n-1)! \left[\frac{n-k}{k!(n-k)!} + \frac{k}{k!(n-k)!} \right]$$

$$= \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

2.6 Cross-polytope Numbers

The cross polytope β_n is the regular polytope in n dimensions corresponding to the convex hull of the points formed by permuting the coordinates $(\pm 1, 0, 0, ..., 0)$.



Definition The Cross Polytope numbers $T(m,n)$ is defined by $T(m,1) = 1$, $T(1,n) = n$, and

$$T(m,n) = T(m-1,n) + T(m,n-1) + T(m-1,n-1)$$

for $m, n \geq 2$ (see [10]). The standard closed form is

$$T(m,n) = \sum_{k=0}^{m-1} \binom{m-1}{k} \binom{n+k}{m}.$$

The following table shows the Cross Polytope numbers $T(m,n)$.

$T(m,n)$	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	1	4	9	16	25	36	49	64	81	
3	1	6	19	44	85	146	231	344		
4	1	8	33	96	225	456	833			
5	1	10	51	180	501	1182				
6	1	12	73	304	985					
7	1	14	99	476						
8	1	16	129							
9	1	18								
10	1									

3 Another way to count n choose k

In [4], There is a new formula for ”n choose k”, it is defined as

$$\binom{n}{k} = \binom{2n+k+1}{k} + \sum_{j=1}^{k} (-1)^j \frac{n+k-j}{j-1} \binom{n+k-j}{j-1} \binom{2n+k+1-2j}{k-j}$$

The net count combinations in which the smallest bad elements are consecutive, with the first ”bad” element between $n+1$ and $n+k$ is

$$\sum_{l=0}^{k-2} \binom{n}{k-l-2} \sum_{m=0}^{k-1} (-1)^m \binom{l+1}{m} \binom{n+k-1-m}{l}.$$

This expression is zero, which is shown in [4] as the inner sum

$$\sum_{m=0}^{l} (-1)^m \binom{l}{m} \binom{n-m}{l-1} = 0$$

The sum of the even (positive) terms equals to the absolute value of the sum of the odd (negative) terms. We denote the sum of even terms as "E_1", and the sum of odd terms as "O_1", we will talk about more in the next section.

4 E and O

Floor Funtion The floor function $\lfloor x \rfloor$, also called the greatest integer function or integer value (Spanier and Oldham 1987), gives the largest integer less than or equal to x.

Definition The even sum and odd sum E_1 and O_1 is defined as the following in [4]

$$E_1(m,n) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{m-2j}{n-1} \binom{n}{2j}$$

$$O_1(m,n) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{m-2j-1}{n-1} \binom{n}{2j+1}$$

From the combinatorial expressions of E_1 and O_1, we can see that they are counting the different numbers, but in fact, they are equal.

Here is the an *example*

$$E_1(6,4) = \sum_{m=0}^{2} \binom{6-2m}{3} \binom{4}{2m} = \binom{6}{3}\binom{4}{0} + \binom{4}{3}\binom{4}{2} + \binom{2}{3}\binom{4}{4} = 20 + 24 = 44$$

$$O_1(6,4) = \sum_{m=0}^{1} \binom{5-2m}{3} \binom{4}{2m+1} = \binom{5}{3}\binom{4}{1} + \binom{3}{3}\binom{4}{3} = 40 + 4 = 44$$

Now, I will give a combinatorial explanation of E_1 and O_1:

For $E_1(m,n)$, we define the committee size of $n-1$, and we have n possible officers, and the number of officers is even.

For $O_1(m,n)$, we define the committee size of $n-1$, and we have n possible officers, and the number of officers is odd.

$E_1(6,4)$: We have 6 people in total $(1,2,3,4,5,6)$, the committee size is 3, we have 4 possible officers. $(3,4,5,6)$ are the possible officers. Since the officers number is even, I can have 2 officers or 0 officer. The total combinations is shown below.

committee	2 officers
125 126 156 256	34
124 126 146 246	35
124 125 146 246	36

committee	2 officers
123 126 136 236	45
123 125 135 235	46
123 124 134 234	56

committee	0 officer
123 124 125 126 134 135 136 145 146 156 234 235 236 245 246 256 345 346 356 456	

E(6,4)=4x6+20=44

$O_1(6,4)$: We have 6 people in total $(1,2,3,4,5,6)$, the committee size is 3, we have 4 possible officers. $(3,4,5,6)$ are the possible officers. Since the officers number is odd, I can have 3 officers or 1 officer. The total combinations is shown below.

committee	1 officer	committee	1 officer
124 125 126 145 146 156 245 246 256 456	3	123 124 125 126 134 136 146 234 236 246	5
123 125 126 135 136 156 235 236 256 356	4	123 124 125 134 135 145 234 235 245 345	6

committee	3 officers
126	345
125	346
124	356
123	456

O(6,4)=4x10+4=44

We can see that E_1 and O_1 counts different things but in fact the results are the same.

Theorem ([4]) For $n \geq 1$and $1 \leq l \leq n$,

$$E_1(m, n) = O_1(m, n) = T(m - n + 1, n).$$

The following table shows the numbers that generate by the definition of E_1.

$E_1(m,n)$	1	2	3	4	5	6	7	8	9
1	1	0							
2	1	2	1						
3	1	4	3	0					
4	1	6	9	4	1				
5	1	8	19	16	5	0			
6	1	10	33	44	25	6	1		
7	1	12	51	96	85	36	7	0	
8	1	14	73	180	225	146	49	8	1
9	1	16	99	304	501	456	231	64	9
10	1	18	129	476	985	1182	833	344	81

We noticed that these numbers are exactly the same as Cross Polytope numbers. Here is the comparison. The difference is E_1 shift the position of T in the table.

$T(m,n)$	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	1	4	9	16	25	36	49	64	81	
3	1	6	19	44	85	146	231	344		
4	1	8	33	96	225	456	833			
5	1	10	51	180	501	1182				
6	1	12	73	304	985					
7	1	14	99	476						
8	1	16	129							
9	1	18								
10	1									

The recursion is obvious from the observation of the table, but we need a proof, and it will be discussed later.

Then we have recurrence formula for both E_1 and O_1

$$E_1(m,n) = E_1(m-1,n-1) + E_1(m-2,n-1) + E_1(m-1,n)$$

$$O_1(m,n) = O_1(m-1,n-1) + O_1(m-2,n-1) + O_1(m-1,n)$$

More generally E_k and O_k are defined and similar to E_1 and O_1, let $k \geq 1$:

$$E_k(m,n) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{m-2j}{n-k} \binom{n}{2j}$$

$$O_k(m,n) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{m-2j-1}{n-k} \binom{n}{2j+1}$$

The algebraic proofs of the recurrence formulas will be discussed in the next section, more details about E_k and O_k will be discussed later.

5 Recurrence

In section 2, we talked about the Fibonacci numbers which follows the recurrence $f_n = f_{n-1} + f_{n-2}$. It is the simplest example of recursive sequence. In section 4, I mentioned the recurrence formulas for E_k and O_k. In [4], there is a combinatorial proof for the recursion. Now, let's talk about the algebraic proofs here.

Pascal's formula is a recursion formula. We can prove the recurrence formula for E_k and O_k algebraically by applying Pascal's formula:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

To begin with the proof, we have to suppose $n \geq 3$, and $k+1 \leq m \leq n$.

We know that

$$E_k(n,m) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} = \binom{m}{2j}\binom{n-2j}{m-k}$$

and

$$O_k(n,m) = \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} = \binom{m}{2j+1}\binom{n-(2j+1)}{m-k}.$$

Consider two cases for m:

Case I: m is odd,

we have$\lfloor \dfrac{m-1}{2} \rfloor = \lfloor \dfrac{m}{2} \rfloor$

We want to prove $E_k(n-1,m) + E_k(n-1,m-1) + E_k(n-2,m-1) = E_k(n,m)$. Using the definition of E_k, we have

$$L.H.S = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2j}\binom{n-1-2j}{m-k} + \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2j}\binom{n-1-2j}{m-1-k} + \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2j}\binom{n-2-2j}{m-1-k}$$

since $\lfloor \dfrac{m-1}{2} \rfloor = \lfloor \dfrac{m}{2} \rfloor$

$$L.H.S = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2j}\binom{n-1-2j}{m-k} + \binom{m-1}{2j}\binom{n-1-2j}{m-1-k} + \binom{m-1}{2j}\binom{n-2-2j}{m-1-k}$$

Apply Pascal's formula for $\binom{m-1}{2j}$,

$$=\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2j}\binom{n-1-2j}{m-k}+\left[\binom{m}{2j}-\binom{m-1}{2j-1}\right]\binom{n-1-2j}{m-1-k}+\binom{m-1}{2j}\binom{n-2-2j}{m-1-k}$$

use distributive law to combine the like terms,

$$=\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2j}\left[\binom{n-2j-1}{m-k}+\binom{n-2j-1}{m-k-1}\right]-\binom{m-1}{2j-1}\binom{n-2j-1}{m-k-1}+\binom{m-1}{2j}\binom{n-2-2j}{m-1-k}$$

$$=\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2j}\binom{n-2j}{m-k}-\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m-1}{2j-1}\binom{n-2j-1}{m-k-1}+\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m-1}{2j}\binom{n-2-2j}{m-1-k}$$

Since $\lfloor\dfrac{m-1}{2}\rfloor=\lfloor\dfrac{m}{2}\rfloor$, substitute $\lfloor\frac{m}{2}\rfloor$ with $\lfloor\frac{m-1}{2}\rfloor$ for the second and the third adder.

Then,

$$L.H.S=\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}\binom{m}{2j}\binom{n-2j}{m-k}-\sum_{j=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m-1}{2j-1}\binom{n-2j-1}{m-k-1}+\sum_{j=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m-1}{2j}\binom{n-2-2j}{m-1-k}$$

For the second term, when $j=0$, $\binom{m-1}{2j-1}=0$, since $2j-1$ is negative.

Let's re-index j, let $j=j+1$, we know that $\lfloor\frac{m-2}{2}\rfloor=\lfloor\frac{m-1}{2}\rfloor-1$, then

$$\sum_{j=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m-1}{2j-1}\binom{n-2j-1}{m-k-1}=\sum_{j=j+1}^{\lfloor\frac{m-1}{2}\rfloor-1}\binom{m-1}{2(j+1)-1}\binom{n-2(j+1)-1}{m-k-1}$$

$$=\sum_{j=j+1}^{\lfloor\frac{m-2}{2}\rfloor}\binom{m-1}{2j+1}\binom{n-2j-3}{m-1-k}=\sum_{j=0}^{\lfloor\frac{m-2}{2}\rfloor}\binom{(m-1)}{2j+1}\binom{(n-2)-2j-1}{(m-1)-k}=O_k(n-2,l-1)$$

Thus,

$$L.H.S = E_k(n,m) - O_k(n-2,m-1) + E_k(n-2,m-1)$$

$$= E_k(n,m) = R.H.S$$

Case II: m is even Then $\lfloor \dfrac{m}{2} \rfloor = \lfloor \dfrac{m-1}{2} \rfloor + 1$, $m-1$ is odd.

We want to prove

$$E_k(n-1,m-1) + E_k(n-2,m-1) + O_k(n-1,m) = O(n,m)$$

The reason we choose E for $m-1$and O for m is because we want them all sum up to $\lfloor \frac{m-1}{2} \rfloor$.

Apply formulas of E and O for the left hand side,

$$\sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2j}\binom{n-1-2j}{m-1-k} + \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2j}\binom{n-2-2j}{m-1-k} + \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2j+1}\binom{n-1-(2j+1)}{m-k}$$

Since they all have the same max value of j, we write the sums together as one sum, and use Pascal's formula for $\binom{m-1}{2j}$ in the second term,

$$\sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2j}\binom{n-1-2j}{m-1-k} + \left[\binom{m}{2j+1} - \binom{m-1}{2j+1}\right]\binom{n-2-2j}{m-1-k} + \binom{m}{2j+1}\binom{n-1-(2j+1)}{m-k}$$

Recombine like terms,

$$\sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2j}\binom{n-1-2j}{m-1-k} + \binom{m}{2j+1}\left[\binom{n-2-2j}{m-1-k} + \binom{n-2-2j}{m-k}\right] - \binom{m-1}{2j+1}\binom{n-2-2j}{m-1-k}$$

$$= \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2j}\binom{n-1-2j}{m-1-k} + \binom{m}{2j+1}\binom{n-1-2j}{m-k} - \binom{m-1}{2j+1}\binom{n-2-2j}{m-1-k}$$

$$= \sum_{j=0}^{\lfloor\frac{m-1}{2}\rfloor} \binom{(m-1)}{2j}\binom{(n-1)-2j}{(m-1)-k} + \sum_{j=0}^{\lfloor\frac{m-1}{2}\rfloor} \binom{m}{2j+1}\binom{n-1-2j}{m-k} - \sum_{j=0}^{\lfloor\frac{m-1}{2}\rfloor} \binom{m-1}{2j+1}\binom{n-2-2j}{m-1-k}$$

For the last term, since m is even, then $\lfloor\frac{m-2}{2}\rfloor = \lfloor\frac{m-1}{2}\rfloor$ Therefore,

$$\sum_{j=0}^{\lfloor\frac{m-1}{2}\rfloor} \binom{(m-1)}{2j}\binom{(n-1)-2j}{(m-1)-k} + \sum_{j=0}^{\lfloor\frac{m-1}{2}\rfloor} \binom{m}{2j+1}\binom{n-2j-1}{m-k} - \sum_{j=0}^{\lfloor\frac{m-2}{2}\rfloor} \binom{(m-1)}{2j+1}\binom{n-1-(2j+1)}{(m-1)-k}$$

$$= E_k(n-1, m-1) + O_k(n, m) - O_k(n-1, m-1)$$

$$= O_k(n, m) = R.H.S$$

6 E_k and O_k

6.1

More generally, E_k and O_k are defined and similar to E_1 and O_1, let $k \geq 1$:

$$E_k(m, n) = \sum_{j=0}^{\lfloor\frac{n}{2}\rfloor} \binom{m-2j}{n-k}\binom{n}{2j}$$

$$O_k(m, n) = \sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor} \binom{m-2j-1}{n-k}\binom{n}{2j+1}$$

They also follow the same recurrence as E_1 and O_1.

$$E_k(m, n) = E_k(m-1, n-1) + E_k(m-2, n-1) + E_k(m-1, n)$$

$$O_k(m, n) = O_k(m-1, n-1) + O_k(m-2, n-1) + O_k(m-1, n)$$

Tables of E_k

E_1(n,m)	1	2	3	4	5
1	1	0	3	-16	85
2	1	2	1	-4	25
3	1	4	3	0	5
4	1	6	9	4	1
5	1	8	19	16	5
6	1	10	33	44	25
7	1	12	51	96	85

E_2(n,m)	2	3	4	5	6
1	2	-2	12	-60	310
2	2	2	4	-20	110
3	2	6	4	-4	30
4	2	10	12	4	6
5	2	14	28	20	6
6	2	18	52	60	30
7	2	22	84	140	110

E_3(n,m)	3	4	5	6	7
1	4	-8	40	-200	1036
2	4	0	16	-80	420
3	4	8	8	-24	140
4	4	16	16	0	36
5	4	24	40	24	12
6	4	32	80	80	36
7	4	40	136	200	140

E_4(n,m)	4	5	6	7	8
1	8	-24	120	-616	3248
2	8	-8	56	-280	1456
3	8	8	24	-104	560
4	8	24	24	-24	176
5	8	40	56	24	48
6	8	56	120	104	48
7	8	72	216	280	176

E_5(n,m)	5	6	7	8	9
1	16	-64	336	-1792	9696
2	16	-32	176	-896	4704
3	16	0	80	-384	2016
4	16	32	48	-128	736
5	16	64	80	0	224
6	16	96	176	128	96
7	16	128	336	384	224

E_6(n,m)	6	7	8	9
1	32	-160	896	-4992
2	32	-96	512	-2688
3	32	-32	256	-1280
4	32	32	128	-512
5	32	96	128	-128
6	32	160	256	128
7	32	224	512	512

6.2

E_{-k} and O_{-k} are defined:

$$E_{-k}(m,n) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{m-2j}{n+k} \binom{n}{2j}$$

$$O_{-k}(m,n) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{m-2j-1}{n+k} \binom{n}{2j+1}$$

But this time, the recurrence is different than before

$$E_{-k}(m,n) = E_{-k}(m-1,n-1) + E_{-k}(m-2,n-1) + E_{-k}(m-1,n) - \binom{n-m-1}{k}$$

$$O_{-k}(m,n) = O_{-k}(m-1,n-1) + O_{-k}(m-2,n-1) + O_{-k}(m-1,n) + \binom{n-m-1}{k}$$

Proof

Pascal's formula has been used again:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

First, we want to prove

$$E_{-k}(n,m) = E_{-k}(n-1,m) + E_{-k}(n-1,m-1) + E_{-k}(n-2,m-1) - \binom{n-m-1}{k}$$

Consider two cases for m:

Case I: m is odd,

we have$\lfloor \frac{m-1}{2} \rfloor = \lfloor \frac{m}{2} \rfloor$

$$E_{-k}(n-1,m) + E_{-k}(n-1,m-1) + E_{-k}(n-2,m-1)$$

$$= \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2j}\binom{n-1-2j}{m+k} + \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2j}\binom{n-1-2j}{m-1+k} + \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2j}\binom{n-2-2j}{m-1+k}$$

$$= \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2j}\binom{n-1-2j}{m+k} + \binom{m-1}{2j}\binom{n-1-2j}{m-1+k} + \binom{m-1}{2j}\binom{n-2-2j}{m-1+k}$$

Apply Pascal's formula for $\binom{m-1}{2j}$,

$$= \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2j}\binom{n-1-2j}{m+k} + \left[\binom{m}{2j} - \binom{m-1}{2j-1}\right]\binom{n-1-2j}{m-1+k} + \binom{m-1}{2j}\binom{n-2-2j}{m-1+k}$$

$$= \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2j}\left[\binom{n-2j-1}{m+k} + \binom{n-2j-1}{m+k-1}\right] - \binom{m-1}{2j-1}\binom{n-2j-1}{m-1+k} + \binom{m-1}{2j}\binom{n-2-2j}{m-1+k}$$

$$= \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2j}\binom{n-2j}{m+k} - \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2j-1}\binom{n-2j-1}{m-1+k} + \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2j}\binom{n-2-2j}{m-1+k}$$

For the second term, replace j by $j+1$, then we have

$$\sum_{j=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{m-1}{2j-1}\binom{n-2j-1}{m-1+k}=\sum_{j+1=0}^{\lfloor\frac{m-1}{2}\rfloor-1}\binom{m-1}{2(j+1)-1}\binom{n-2(j+1)-1}{m-1+k}$$

$$=\sum_{j+1=0}^{\lfloor\frac{m-2}{2}\rfloor}\binom{m-1}{2j+1}\binom{n-2-2j-1}{m-1+k}$$

Then

$$L.H.S=E_{-k}(n,m)-O_{-k}(n-2,m-1)+E_{-k}(n-2,m-1)$$

By previous Theorem in [8], we have

$$E_{-k}(n-2,m-1)-O_{-k}(n-2,m-1)=\binom{(n-2)-(m-1)}{k}=\binom{n-m-1}{k}$$

Thus

$$L.H.S=E_{-k}(n,m)+\binom{n-m-1}{k}$$

Hence

$$E_{-k}(n,m)=E_{-k}(n-1,m)+E_{-k}(n-1,m-1)+E_{-k}(n-2,m-1)-\binom{n-m-1}{k}$$

Case II: m is even, then $\lfloor\frac{m}{2}\rfloor=\lfloor\frac{m-1}{2}\rfloor+1$, $m-1$ is odd.

$$E_{-k}(n-1,m)+E_{-k}(n-1,m-1)+E_{-k}(n-2,m-1)$$

$$= \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2j}\binom{n-1-2j}{m+k} + \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2j}\binom{n-1-2j}{m-1+k} + \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2j}\binom{n-2-2j}{m-1+k}$$

$$= \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2j}\binom{n-1-2j}{m+k} + \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-1}{2j}\binom{n-1-2j}{m-1+k} + \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-1}{2j}\binom{n-2-2j}{m-1+k}$$

$$-\binom{m-1}{m}\binom{n-1-m}{m+k-1} - \binom{m-1}{m}\binom{n-2-m}{m+k-1}$$

$$= \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2j}\binom{n-1-2j}{m+k} + \binom{m-1}{2j}\binom{n-1-2j}{m-1+k} + \binom{m-1}{2j}\binom{n-2-2j}{m-1+k} - 0 - 0$$

$$= \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2j}\binom{n-1-2j}{m+k} + \left[\binom{m}{2j} - \binom{m-1}{2j-1}\right]\binom{n-1-2j}{m-1+k} + \binom{m-1}{2j}\binom{n-2-2j}{m-1+k}$$

$$= \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2j}\left[\binom{n-2j-1}{m+k} + \binom{n-2j-1}{m+k-1}\right] - \binom{m-1}{2j-1}\binom{n-2j-1}{m-1+k} + \binom{m-1}{2j}\binom{n-2-2j}{m-1+k}$$

$$= \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m}{2j}\binom{n-2j}{m+k} - \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-1}{2j-1}\binom{n-2j-1}{m-1+k} + \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor+1} \binom{m-1}{2j}\binom{n-2-2j}{m-1+k}$$

For the second term, replace j by $j+1$, then we have

$$\sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-1}{2j-1}\binom{n-2j-1}{m-1+k} = \sum_{j+1=0}^{\lfloor \frac{m}{2} \rfloor-1} \binom{m-1}{2(j+1)-1}\binom{n-2(j+1)-1}{m-1+k}$$

$$= \sum_{j+1=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2j+1}\binom{n-2-2j-1}{m-1+k}$$

Since $\lfloor \frac{m-1}{2} \rfloor = \lfloor \frac{m-2}{2} \rfloor$, then

$$\sum_{j+1=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2j+1}\binom{n-2-2j-1}{m-1+k} = \sum_{j+1=0}^{\lfloor \frac{m-2}{2} \rfloor} \binom{m-1}{2j+1}\binom{n-2-2j-1}{m-1+k} = O_{-k}(n-2, m-1)$$

For the last term,

$$\sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor +1} \binom{m-1}{2j}\binom{n-2-2j}{m-1+k} + \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2j}\binom{n-2-2j}{m-1+k} = E_{-k}(n-2, m-1)$$

Then

$$L.H.S = E_{-k}(n, m) - O_{-k}(n-2, m-1) + E_{-k}(n-2, m-1)$$

Hence

$$E_{-k}(n, m) = E_{-k}(n-1, m) + E_{-k}(n-1, m-1) + E_{-k}(n-2, m-1) - \binom{n-m-1}{k}$$

Next, we want to prove

$$O_{-k}(n,m) = O_{-k}(n-1,m) + O_{-k}(n-1,m-1) + O_{-k}(n-2,m-1) + \binom{n-m-1}{k}$$

Similar to the proof of the recursion for E_{-k}

Consider two cases for m:

Case I: m is even, then$\lfloor \frac{m-1}{2} \rfloor = \lfloor \frac{m-2}{2} \rfloor$

$$O_{-k}(n-1,m) + O_{-k}(n-1,m-1) + O_{-k}(n-2,m-1)$$

$$= \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2j+1}\binom{n-2j-2}{m+k} + \sum_{j=0}^{\lfloor \frac{m-2}{2} \rfloor} \binom{m-1}{2j+1}\binom{n-2j-2}{m-1+k} + \sum_{j=0}^{\lfloor \frac{m-2}{2} \rfloor} \binom{m-1}{2j+1}\binom{n-2j-3}{m-1+k}$$

$$= \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2j+1}\binom{n-2j-2}{m+k} + \binom{m-1}{2j+1}\binom{n-2j-2}{m-1+k} + \binom{m-1}{2j+1}\binom{n-2j-3}{m-1+k}$$

$$= \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2j+1}\binom{n-2j-2}{m+k} + \left[\binom{m}{2j+1} - \binom{m-1}{2j}\right]\binom{n-2j-2}{m-1+k} + \binom{m-1}{2j+1}\binom{n-2j-3}{m-1+k}$$

$$= \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2j+1}\left[\binom{n-2j-2}{m+k} + \binom{n-2j-2}{m-1+k}\right] - \binom{m-1}{2j}\binom{n-2j-2}{m-1+k} + \binom{m-1}{2j+1}\binom{n-2j-3}{m-1+k}$$

$$= \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2j+1}\binom{n-2j-1}{m+k} - \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2j}\binom{n-2j-2}{m-1+k} + \sum_{j=0}^{\lfloor \frac{m-2}{2} \rfloor} \binom{m-1}{2j+1}\binom{n-2j-3}{m-1+k}$$

$$= O_{-k}(n,m) - E_{-k}(n-2,m-1) + O{-}k(n-2,m-1)$$

By previous Theorem (number), we have

$$E_{-k}(n-2,m-1) - O_{-k}(n-2,m-1) = \binom{(n-2)-(m-1)}{k} = \binom{n-m-1}{k}$$

Then

$$L.H.S = O_{-k}(n,m) - \binom{n-m-1}{k}$$

Hence

$$O_{-k}(n,m) = O_{-k}(n-1,m) + O_{-k}(n-1,m-1) + O_{-k}(n-2,m-1) + \binom{n-m-1}{k}$$

Case II: m is odd, then $\lfloor \frac{m-1}{2} \rfloor = \lfloor \frac{m-2}{2} \rfloor + 1$

$$O_{-k}(n-1,m) + O_{-k}(n-1,m-1) + O_{-k}(n-2,m-1)$$

$$= \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2j+1}\binom{n-2j-2}{m+k} + \sum_{j=0}^{\lfloor \frac{m-2}{2} \rfloor} \binom{m-1}{2j+1}\binom{n-2j-2}{m-1+k} + \sum_{j=0}^{\lfloor \frac{m-2}{2} \rfloor} \binom{m-1}{2j+1}\binom{n-2j-3}{m-1+k}$$

$$= \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2j+1}\binom{n-2j-2}{m+k} + \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2j+1}\binom{n-2j-2}{m-1+k} - \binom{m-1}{m}\binom{n-m-1}{m-1+k}$$

$$+ \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2j+1}\binom{n-2j-3}{m-1+k} - \binom{m-1}{m}\binom{n-m-2}{m-1+k}$$

$$= \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2j+1}\binom{n-2j-2}{m+k} + \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2j+1}\binom{n-2j-2}{m-1+k} - 0 + \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2j+1}\binom{n-2j-3}{m-1+k} - 0$$

$$= \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2j+1}\binom{n-2j-2}{m+k} + \left[\binom{m}{2j+1} - \binom{m-1}{2j}\right]\binom{n-2j-2}{m-1+k} + \binom{m-1}{2j+1}\binom{n-2j-3}{m-1+k}$$

$$= \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2j+1}\left[\binom{n-2j-2}{m+k} + \binom{n-2j-2}{m-1+k}\right] - \binom{m-1}{2j}\binom{n-2j-2}{m-1+k} + \binom{m-1}{2j+1}\binom{n-2j-3}{m-1+k}$$

$$= \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m}{2j+1}\binom{n-2j-1}{m+k} - \sum_{j=0}^{\lfloor \frac{m-1}{2} \rfloor} \binom{m-1}{2j}\binom{n-2j-2}{m-1+k} + \sum_{j=0}^{\lfloor \frac{m-2}{2} \rfloor} \binom{m-1}{2j+1}\binom{n-2j-3}{m-1+k}$$

$$= O_{-k}(n,m) - E_{-k}(n-2,m-1) + O{-}k(n-2,m-1)$$

Similarly to case I,

$$O_{-k}(n,m) = O_{-k}(n-1,m) + O_{-k}(n-1,m-1) + O_{-k}(n-2,m-1) + \binom{n-m-1}{k}$$

7 Reflection

From the observation from E_k tables, we can see the reflections but the equations is not obvious to conclude. So I made several calculations and comparisons and finally came up the reflection theorem and the algebraic proof.

Theorem

$$E_k(n,m) = (-1)^{m-k} E_k(-n+2m-k-1,m)$$

proof

Knowing that

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$$

$$\binom{n}{k} = \binom{n}{n-k}$$

Then

$$E_k(n,m) = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} \binom{n-2j}{m-k}\binom{m}{2j} = \sum_{j=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^{m-k} \binom{-n+2j+m-k-1}{m-k}\binom{m}{m-2j}$$

Suppose m is even, let's re-index j now: Let $l = \frac{m}{2} - j$, then $2l = m - 2j$ and $2j = m - 2l$.

Then

$$E_k(n,m) = (-1)^{m-k} \sum_{l=0}^{\lfloor \frac{m}{2} \rfloor} \binom{-n+(m-2l)+m-k-1}{m-k}\binom{m}{2l}$$

$$= (-1)^{m-k}\sum_{l=0}^{\lfloor\frac{m}{2}\rfloor}\binom{(-n+2m-k-1)-2l}{m-k}\binom{m}{2l}$$

We substitute l with j, then

$$E_k(n,m) = (-1)^{m-k}\sum_{j=0}^{\lfloor\frac{m}{2}\rfloor}\binom{(-n+2m-k-1)-2j}{m-k}\binom{m}{2j}$$

Obviously, this is equal to $E_k(-n+2m-k-1,m)$. Thus

$$E_k(n,m) = (-1)^{m-k}E_k(-n+2m-k-1,m)$$

Now let's prove the the case of m is odd, then $m-1$ is even, we use $O_k(n,m)$ to prove the reflection

$$O_k(n,m) = \sum_{j=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{n-2j-1}{m-k}\binom{m}{2j+1}$$

$$= \sum_{j=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^{m-k}\binom{(-n+2j+1)+(m-k)-1}{m-k}\binom{m}{m-2j-1}$$

Let $l = \frac{m-1}{2} - j$, then $2l = m-1-2j$, $2j = m-1-2l$, Substitute $2j$, then

$$O_k(n,m) = \sum_{l=0}^{\lfloor\frac{m-1}{2}\rfloor}(-1)^{m-k}\binom{-n+(m-1-2l)+1+(m-k)-1}{m-k}\binom{m}{2l}$$

Substitute l with j, then

$$O_k(n,m) = (-1)^{m-k}\sum_{j=0}^{\lfloor\frac{m-1}{2}\rfloor}\binom{-n+2m-k-1-2j}{m-k}\binom{m}{2j}$$

$$= (-1)^{l-1}E_k(-n+2m-k-1,m)$$

8 Generalizations of Cross-polytope Numbers

Theorem:

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}E_k(n+i,k+m-1)=0$$

Proof:

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}E_k(n+i,k+m-1)$$

$$=\sum_{i=0}^{m}(-1)^i\binom{m}{i}\sum_{j=0}^{\lfloor\frac{k+m-1}{2}\rfloor}\binom{k+m-1}{2j}\binom{n+i-2j}{m-1}$$

$$=\sum_{i-0}^{m}\sum_{j=0}^{\lfloor\frac{k+m-1}{2}\rfloor}(-1)^i\binom{m}{i}\binom{k+m-1}{2j}\binom{n+1-2j}{m-1}$$

$$=\sum_{j=0}^{\lfloor\frac{k+m-1}{2}\rfloor}\binom{k+m-1}{2j}\sum_{i=0}^{m}(-1)^i\binom{m}{i}\binom{n+i-2j}{m-1}$$

By a theorem in [4]:

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}\binom{n-i}{m-1}=0$$

Thus

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}\binom{n+i-2j}{m-1}=0$$

Therefore

$$\sum_{i=0}^{m}(-1)^i\binom{m}{i}E_k(n+i,k+m-1)=0$$

So if

$$a_n=E_k(n,k+m-1)$$

then

$$a_n = s\sum_{i=1}^{n}(-1)^{i+1}\binom{m}{i}a_{n-1}$$

$E_0 + O_0$ are Delannoy Numbers, which will be introduced in the next section.

We noticed that E_k is devisable by 2^{k-1}, the following are the tables of $\frac{E_k}{2^{k-1}}$

E_1	1	2	3	4	5	6	7	8	9
1	1	0							
2	1	2	1						
3	1	4	3	0					
4	1	6	9	4	1				
5	1	8	19	16	5	0			
6	1	10	33	44	25	6	1		
7	1	12	51	96	85	36	7	0	
8	1	14	73	180	225	146	49	8	1
9	1	16	99	304	501	456	231	64	9
10	1	18	129	476	985	1182	833	344	81

$\frac{1}{2}E_2$	2	3	4	5	6	7
1	1					
2	1	1				
3	1	3	2			
4	1	5	6	2		
5	1	7	14	10	3	
6	1	9	26	30	15	3
7	1	11	42	70	55	21

$\frac{1}{4}E_3$	3	4	5	6	7	8	9	10	11
1	1								
2	1	0							
3	1	2	2						
4	1	4	4	0					
5	1	6	10	6	3				
6	1	8	20	20	9	0			
7	1	10	34	50	35	12	4		
8	1	12	52	104	105	56	16	0	
9	1	14	74	190	259	196	84	20	5
10	1	16	100	316	553	560	336	120	25

$\frac{1}{8}E_4$	4	5	6	7	8	9
2	1					
3	1	1				
4	1	3	3			
5	1	5	7	3		
6	1	7	15	13	6	
7	1	9	27	35	22	
8	1	11	43	77	70	

$\frac{1}{16}E_5$	5	6	7	8	9	10
2	1					
3	1	0				
4	1	2	3			
5	1	4	5	0		
6	1	6	11	8	6	
7	1	8	21	24	14	0
8	1	13	35	56	46	20

$\frac{1}{32}E_6$	6	7	8	9
3	1			
4	1	1		
5	1	3	4	
6	1	5	8	4
7	1	7	16	16

9 Delannoy Numbers

[9]In mathematics, a Delannoy number D describes the number of paths from the southwest corner $(0, 0)$ of a rectangular grid to the northeast corner (m, n), using only single steps north, northeast, or east. The Delannoy numbers are named after French army officer and amateur mathematician

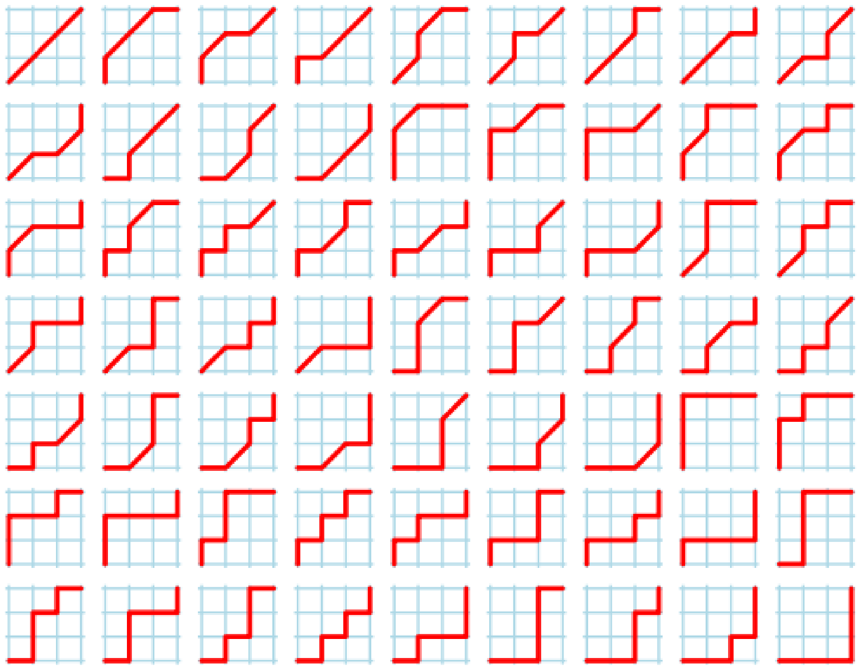
Henri Delannoy.

Also, diagonal of array is defined by

$$m(0,1) = m(1,0) = 1,$$

$$m(i,j) = m(i-1,j-1) + m(i-1,j) + m(i,j-1).$$

The Delannoy number $D(3,3)$ equals 63. The following figure illustrates the 63 Delannoy paths through a 3 × 3 grid:



In [3], Delannoy counts the total number of paths to (x, y) as follows:

Each diagnal step e is equivalent to a vertical step a and a horizontal step b. If the Queen moves from $(0,0)$ to (x,y) with z diagonal steps e, then the number of such walks corresponds to the number of permutations with $x-z$ letters a, $y-z$ letters b, and z letters e, which is equal to

$$\frac{(y+x-z)!}{(y-z)!(x-z)!z!} = \binom{x+y-2z}{x-z}\binom{x+y-z}{z} = \binom{x+y-z}{x}\binom{x}{z}.$$

The total number of paths to (x, y) will be given by

$$\sum_{z=0}^{x} \binom{x}{z} \binom{x+y-z}{x},$$

if $x < y$, in other case, one permutes x and y in this expression, which also has the form

$$\sum_{z=0}^{x} 2^z \binom{y}{z} \binom{x}{z}$$

The Delannoy and Cross Polytope numbers are members of the same family. They are both satisfy the same recurrence.

10 Conclusion

Mathematical truths are given in theorems and the most interesting and exciting part for people who do research in math is to came up new mathematical formulas or new proofs of the existing theorems. The recursion formula of the cross polytope number in the table of E_1 is obvious by observation, there is a combinatorial proof in [4], but I came up an algebraic proof. So by breaking the "whole piece" into parts and recombine the pieces into whole by using Pascal's identity is the main procedure for the algebraic proof.

A lot of mathematical theorems come from observations. But the hardest the part is proving. In order to proof the new theorem, sometimes we need to take advantage of the existing evidence to support our new "truth". There might be serval ways to solve the same questions. At the undergraduate level, I found the easiest way for me to understand a proof is follow the steps the author has given in his paper. So that's why I want to share with the reader with my algebraic proofs for existing formulas.

From all the definitions of E_k and O_k, is it hard to see directly for the recursion and reflection identities. But when we write out the numbers, we can see the recursion and reflection through the tables of E_k and O_k. Reflection shows important character of the numbers that is not obvious either from the definition or the recursion.

Delannoy numbers are probably the oldest useful doubly-recursive sequence. They count the Lattice Paths. E_k, O_k, E_{-k}, O_{-k} are the generalizations of the Delannoy. The proofs of the recursion is not obvious from the definition. The reflection is not obvious but mathematically interesting to me to seek a formula and prove it.

I think this research experience definitely helps me build my fundamental knowledge in the mathematical fields of combinatorics and number theory. I learned from Dr. Edwards of how to construct a mathematical research and different approaches to seek the algebraic proofs for the existing or newly constructed formulas. It is a good preparation for my future graduate study.

References

[1] Benjamin, Arthur T., and Jennifer J. Quinn. *Proofs that Really Count The Art of Combinatorial Proof*. Washington: Mathematical Association of America, 2014. Print.

Combinatorial Proof. Washington: Mathematical Association of America, 2014. Print. This is a very comprehensive book in discrete math. In chapter 5 and chapter 6, the book introduces binomial identities and alternating sign binomial identities. It is a great contribution to my proofs of the recursion and reflection properties in this project. This book is also a great study material of discrete math. There are also exercise in this book that have solutions attached. As an entry level undergraduate student , I find this book is easier to start with.

[2] Dunlap, Richard A. *The golden ratio and Fibonacci numbers*. New Jersey: World Scientific, 2008. Print.

This is a very interesting book to read for the applications of the golden ratio and Fibonacci number which range from the plant growth and the crystallographic structure of certain solids to the development of computer algorithms for searching data bases. The basic property of the the golden ratio gives me a thorough understanding of the Fibonacci sequence. I am also intrigued by the interesting and beautiful tiling graphs and the real world picture from the plants and creatures.

[3] H. Delannoy, Emploi de L'échiquier pour la Résolution de Divers Problèmes de Probabilité, Assoc. Franc. Bordeaux 24 (1895), 70-90.

This paper is originally written in French. French army officer and amateur mathematician Henri Delannoy first introduced the numbers that describe the number of paths from the southwest corner $(0, 0)$ of a rectangular grid to the northeast corner (m, n), using only single steps north, northeast, or east. I don't know about French, but my advisor Dr. Edwards masters in French, thus he provides me his translated version of this material from Delannoy's paper. The material contains the number of solutions corresponds to the number of permutations with $x - z$ letters a, $y - z$ letters b, and z letters c.

[4] Steven Edwards and William Griffiths, *A Binomial Identity Related to Cross Polytope Numbers*, The Fibonacci Quarterly, Vol. 4, No. 3, (2016), 253-256.

This paper is the main foundation of my project. My advisor Steven Edwards has published a new combinatorial expression $\binom{n}{k}$. One of the discovery of this paper is that the sum of even/odd index terms of the combinatorial expression is actually related to the cross-polytope numbers. In this paper, there are induction proof for the recursion formula that related to the cross-polytope recursion, and the induction proof for the equal relationship of E and O. It helps me build a foundation for my algebraic proofs for the recursion. The new expression for E_k and O_k also based on this paper.

[5] John Bodeen, Steve Butler, Taekyoung Kim, Xiyuan Sun, and Shenzhi Wang, *Tiling a Strip With Triangles*, Electronic Journal of Combinatorics 21 (2014), no. 1.

This paper introduces the tilings of a $2 \times n$ "triangular strip" with triangles. There are several connections between the tilings and the well known sequences such as Fibonacci numbers and Pell numbers. It is a great material that introducing using triangles stripes to proof the the desired recursion. The colored pictures of the triangles strips help readers better visualize and understand the concept of the recursion proofs for sequence numbers.

[6] Wilf, Herbert S. *Generatingfunctionology*. Wellesley, MA: A K Peters, 2006. Print.

This book is about generating functions and some of their uses in discrete mathematics. It is a comprehensive book that contains lots of examples of how to formulate the generating functions, a bridge between discrete mathematics, on the one hand, and continuous analysis on the other. It is a great material to study discrete math since it has tons of examples and exercise problems. It is possible to study them solely as tools for solving discrete problems. As such there is much that is powerful and magical in the way generating functions give unified methods for handling such problems.

[7] Sulanke, Robert A. *Objects Counted by the Central Delannoy Numbers*. Journal of Integer Sequences 6 (2003), Article 03.1.5

This paper summarizes 29 configurations counted by the central Delannoy number d_n, which counts the lattice paths running from $(0,0)$ to (n,n) that use the steps $(1,0)$, $0,1)$, and $(1,1)$. This paper is a good material for me to study counting and combinatorics. The examples that using central Delannoy are well explained in this paper. The second section of this paper invites bijective, recursive, and generating function proofs that the Delannoy numbers do indeed count the configurations.

[8] Steven Edwards and William Griffiths, *Generalizations of Delannoy and Cross Polytope Numbers*, The Fibonacci Quarterly.

[9] OEIS Foundation, Inc. (2011), Square array of Delannoy numbers, `http://oeis.org/A008288`.

[10] OEIS Foundation, Inc. (2011), Table of figurate numbers for the n-dimensional cross polytopes, `http://oeis.org/A142978`.