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A Note on k-price Auctions with Complete Information When Mixed Strategies are Allowed

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A note on k-price auctions with complete information when mixed strategies are allowed*

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Abstract

Restricting attention to players who use pure strategies, Tauman (2001) proves that in a k-price auction ($k \geq 3$) for every Nash equilibrium in which no player uses a weakly dominated strategy: (i) the bidder with the highest value wins the auction and (ii) pays a price higher than the second-highest value among the players, thereby generating more revenue for the seller than would occur in a first- or second-price auction. We show that these results do not necessarily hold when mixed strategies are allowed. In particular, we construct an equilibrium for $k \geq 4$ in which the second-highest valued player wins the auction and makes an expected payment strictly less than her value. This equilibrium–which exists for any generic draw of player valuations–involves only one player using a nondegenerate mixed strategy, for which the amount of mixing can be made arbitrarily small.

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1 Introduction

Tauman (2001) considers k-price auctions, in which a seller of a single unit solicits monetary bids from $n \geq k$ players, with the player submitting the highest bid winning the auction and paying the k-th highest bid (with ties broken randomly). Players have complete information about each other's values and are named such that their values are nonincreasing:

$$v_1 > v_2 > \cdots > v_n.$$

As far as we know Tauman's is the only paper that considers k-price auctions in an complete information setting. The incomplete information setting is studied by Monderer and Tennenholtz (2000, 2004), Azrieli and Levin (2012), and Tumendemberel (2013). Payoffs to players in Tauman's setting are linear: if player i wins a good with probability q and pays m, her payoff is: $qv_i - m$. Tauman restricts strategy spaces to pure strategies and restricts attention to Nash equilibria in which players do not use weakly dominated strategies (he shows that a pure strategy bid b_i' is weakly dominated if and only if $b_i' < v_i$). In this setting, Tauman constructs a pure-strategy Nash equilibrium in which no player uses a weakly dominated strategy and finds that for every such equilibrium, the following results hold:

R1: Player 1 (with the highest value) wins the auction with probability one.

R2: The seller obtains an expected profit π in the interval $[v_2, v_1]$.

In our note, we maintain the complete information setting, but allow the players to use mixed strategies. We construct a Nash equilibrium (in which no player uses a weakly dominated strategy) for the k-price auction ($k \geq 4$) in which neither result R1 nor R2 holds. Our equilibrium exists for all (generic) draws of valuations such that valuations differ. Further, only player k uses a nondegenerate mixed strategy: all other players use a pure strategy. Further still, such equilibria can be constructed in which player k places an arbitrarily close to one probability on a single bid. Thus, our note sheds light on the critical nature of the pure-strategy assumption in Tauman's note.

Before getting underway, we remark that mixed strategies are natural to consider in auction games with complete information. For example, Hirshleifer and Riley (1992) construct mixed strategy Nash equilibria for the first-price auction and Hillman and Riley (1989) do the same for the all-pay auction. In the first-price auction, the second-highest valued player uses a mixed strategy which never wins in equilibrium (but is critical for providing the highest-valued player's equilibrium incentives). In the all-pay auction, both the second-highest and highest-valued bidders use mixed strategies, with the second-highest valued player winning with positive probability. More generally, $k-$price auctions have an all-or-nothing aspect, where all but one player lose the auction and get a payoff of 0 while one player wins the auction and can get a nonzero payoff, making these auctions not so different from zero sum games such as matching pennies where mixed-strategy Nash equilibria obtain. For one example, see Walker and Wooders (2001) who give evidence that tennis players use mixed strategies in deciding where to serve and defend.

2 Results

We construct a Nash equilibrium in mixed strategies for any k-price auction with $k \geq 4$. We first parameterize bid strategies and then show that with appropriately chosen parameter values, the strategies constitute a Nash equilibrium. Let H be a potential bid such that $H > v_1$. Suppose that the players use the following strategies. All players but player k use a pure strategy: player 1 bids her value v_1; player 2 bids $H + \varepsilon$ ($\varepsilon > 0$); players 3 through $k-1$ each bid H; and each player $i > k$ bids her value v_i. Player k uses a mixed strategy: bidding H with probability p and bidding v_3 with probability $1-p$, where $0 < p < 1$. Observe that these strategies are weakly undominated (since no player bids below her value).

We next construct the conditions needed to support the equilibrium. Using the proposed strategies, player 2 will win the auction and will pay v_1 with probability p and pay v_3 with probability $1-p$; all other players will lose the auction and earn payoffs of 0.

If player 1 unilaterally deviates by bidding high enough to win the auction (say by bidding $H+2\varepsilon$), then the k-th highest price will be set by the realization of player k's mixed strategy. Equilibrium requires that player 1's expected payoff from such a deviation be nonpositive:

$$p\left(v_1 - H\right) + (1 - p)\left(v_1 - v_3\right) \leq 0. \tag{1}$$

Any deviation by player 1 such that she still loses the auction leaves her payoff unchanged.

For the proposed strategies to form an equilibrium, player 2 must earn a nonnegative payoff:

$$p\left(v_2 - v_1\right) + (1 - p)\left(v_2 - v_3\right) \geq 0. \tag{2}$$

If condition (2) holds, then there are no profitable unilateral deviations for player 2: any bid higher than H leaves her payoff unchanged; any bid lower than H gives her payoff 0; and a bid of exactly H ties for highest, and she will sometimes get the payoff given in (2) and sometimes 0, depending on the seller's random selection of the winner.

If any of the remaining players ($i > 2$) unilaterally deviates by bidding high enough to win the auction, the resulting price will be either v_1 or v_3, thereby not increasing the player's payoff.

Thus, the conjectured strategies form a Nash equilibrium so long as both conditions (1) and (2) hold. For *any* profile of player values, both conditions (1) and (2) can be satisfied by simultaneously making H large enough and $p > 0$ small enough.[1]

These strategies do not form an equilibrium in a third-price auction ($k = 3$). In this case, there is no bidder that always bids H. By unilaterally deviating to bidding $b' \in (v_1, H)$ with probability one, player 2 will still win the auction when it is profitable for her to do so (when player 3 bids v_3), but player 2 will lose the auction whenever winning would result in a loss for her (when player 3 bids H). Thus, player 2's expected payoff from this deviation strictly increases to $(1 - p)\left(v_2 - v_3\right)$, invalidating the proposed strategies from forming a Nash equilibrium.

[1] Conditions (1) and (2) can respectively be expressed as $H \geq \left(v_1 - v_3\right)/p + v_3$ and $p \leq \left(v_2 - v_3\right)/(v_1 - v_3)$.

Summing up, we have found equilibria for the k-price auction ($k \geq 4$) in which only player k is using a nondegenerate mixed strategy and, further, her probability $1-p$ of bidding v_3 can be made arbitrarily close to one, if in turn H is sufficiently high for condition (1) to hold. Player 2 (with the second highest value) wins the auction with probability one and the seller obtains expected revenue of $pv_1 + (1-p)v_3$. This revenue is strictly less than v_2 for equilibria with $p < (v_2 - v_3)/(v_1 - v_3)$. Consequently, these mixed-strategy Nash equilibria violate both R1 and R2 that Tauman found to be true of all pure-strategy Nash equilibria.

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