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On the L_p -error of approximation of bivariate functions by harmonic splines

Yuliya Babenko · Tatyana Leskevich

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Abstract Interpolation by various types of splines is the standard procedure in many applications. In this paper we shall discuss harmonic spline “interpolation” (on the lines of a grid) as an alternative to polynomial spline interpolation (at vertices of a grid). We will discuss some advantages and drawbacks of this approach and present the asymptotics of the L_p -error for adaptive approximation by harmonic splines.

Keywords interpolation · adaptive · harmonic spline · optimal error · asymptotics

Mathematics Subject Classification (2000) 41A15 · 41A60 · 68W25 · 97N50

1 Harmonic splines as an alternative to polynomial splines

In [5] authors investigated the question of adaptive approximation by polynomial splines on box partitions in \mathbb{R}^d . They obtained a general description as well as sharp constant in several popular cases. However, polynomial splines might not be the best option when it comes to interpolating a function over box partitions. There is an alternative - harmonic splines - which can be viewed as a direct multivariate generalization of linear splines in one dimension. This paper addresses approximation of bivariate functions and, therefore, from now on we shall consider harmonic splines of two variables.

The following several observations help to see the analogy between univariate linear splines and bivariate harmonic splines on rectangular partitions more clear:

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1. Linear univariate splines interpolate the given function on the whole boundary of the mesh which is the points of the partition of the domain. While bilinear splines defined on rectangular partitions interpolate the given function only at the vertices of the mesh elements, not on the whole boundary, harmonic splines are constructed to coincide with the original function on the boundary of the meshes - on the lines (as oppose to just at the points). Litvin in [10] introduces different terminology to distinguish between these types of interpolation and he says that harmonic splines “*interlineates*” function instead of just *interpolating*. Using harmonic splines has advantages and drawbacks and the choice depends on a particular problem. The strong advantage is that it uses more information about the function. However, it also imposes restrictions on the mesh. In the case of square domain (considered in this paper) the meshes will consist of mostly squares. In the case of a rectangular domain, the elements of the mesh will have the same shape (same width to length ratio) as the domain, and only size will adapt to the local behavior of the function.
2. Both univariate linear spline $P_1(x)$ and bivariate harmonic spline $U(x,y)$ satisfy corresponding operator equations. Indeed, if we denote by $\Delta_1 = \frac{d^2}{dx^2}$ and $\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, then clearly

$$\Delta_1 P_1(x) = 0 \quad \text{and} \quad \Delta_2 U(x,y) = 0.$$

3. A univariate linear interpolating spline and bivariate harmonic spline attain maximum and minimum values on the boundary of the element.
4. If the values of the univariate linear spline at internal nodes are determined from the continuity condition on the first derivative, spline degenerates into a segment of straight line. If the values of harmonic spline are determined from the condition of continuity of the normal derivative to the interior boundary of the mesh, then the spline degenerates into a harmonic function on the union of domains (which is, by the way, far less restrictive).
5. Both have very similar local estimates of the error of approximation [8].

For more discussion of properties of harmonic and polyharmonic splines (or polysplines) see the monographs [9, 10] and references therein.

To approximate the bivariate function (so that the approximant also coincides with the given function on some lines) polynomial splines, in particular, blending splines, can also be used. However, there exists a broad class of functions that can be well approximated by harmonic splines, but are not at all approximated by such polynomial splines under the same requirements concerning the order of smoothness. Let us present an example of such a function.

Example ([8]). Let us consider the function

$$f(x,y) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)\alpha_k^2 \sinh(\alpha_k b)} (\sinh(\alpha_k(y-b)) + \sinh(\alpha_k b) - \sinh(\alpha_k y)) \sin(\alpha_k x),$$

$$0 \leq x \leq a, \quad 0 \leq y \leq b, \quad \alpha_k = (2k+1)\pi/a.$$

For this function

$$\lim_{x \rightarrow 0, y \rightarrow 0} \frac{\partial^2 f(x, y)}{\partial x \partial y} = \infty,$$

and, hence, it cannot be approximated by blending polynomial splines. On the other hand, $\Delta f(x, y) = -1$ which implies that f can be well approximated by harmonic splines.

In univariate case the idea of using splines constructed based on solution of some differential operator equations, called L -splines, has been more or less well-developed (see, for example, [15] and references therein). There has been several attempts to generalize this concept to multivariate case. For example, Litvin in his monograph [10] introduces concepts of “interlineation” and “interflotation”. These concepts are natural and direct generalizations of interpolation to the case when one needs to reconstruct the given function based on knowing values of function (and possibly its normal derivatives up to some order) on one or several lines (“interlineation”) or m -dimensional ($1 < m < d$) linear manifolds (“interflotation”) in \mathbb{R}^d .

As already mentioned earlier, in [10] for the given on \mathbb{R}^2 function Litvin suggests and compares two methods of polynomial and non-polynomial “interlineation” on several lines. The first (polynomial) is a generalization of Hermite polynomials which preserves the smoothness order of the original function. The second (not polynomial) operator is a generalization of D’Alambert operator, which is the solution of homogeneous wave equation, to the case when data is given as values of the normal derivatives of all orders up to and including N ($N > 1$).

There exist many applications in which data is given not at a discrete set of points but on some linear manifolds (lines, planes, etc.). One of many examples is cartography. The measurement and analysis of bathymetric measurements is one of the core areas of modern hydrography, and a fundamental component in ensuring the safety of water transportation, efficiency of offshore resource development, precision and effectiveness in recovering underwater objects as well as search and rescue missions. The data used to make bathymetric maps today typically comes from an echosounder (sonar) mounted beneath or over the side of a boat, “pinging” a beam of sound downward at the seafloor or from remote sensing LIDAR or LADAR systems. In all the cases, the data obtained are univariate - along the lines, curves or, more generally, a grid composed of them - depending on the course of the surveyor vessel (which can be adapted when needed). The goal is to recover the function describing the relief of the seafloor based on the “traces” of this function on the lines which described the course of the vessel with sonar.

Similar technology is used for mapping surfaces of cosmic objects. Only instead of sonar, which uses sound waves, radars are used. Radars are based on high-frequency electromagnetic radiation which can penetrate the surroundings of objects, for instance, the clouds covering Venus.

Another example of applications to surface reconstruction based on data given on lines, curves, or hyperplanes would be manufacturing of car, aircraft etc. bodies. Currently, most popular tool used in these models is polynomial splines. Being easy to manipulate, they have many drawbacks: lack of global smoothness (if working with convenient low degree polynomial pieces) and lack of flexibility in choosing the mesh (for instance, it might be necessary to base a part of the body on hexagon or pentagon,

not only triangle or rectangle). Harmonic and their generalization - polyharmonic - splines have great advantage in both directions: they preserve higher smoothness and they can be constructed on any (more or less reasonable) domain.

2 Notation, statement of the main problem, and the main result

Let in the space \mathbb{R}^2 of points $\mathbf{x} = (x_1, x_2)$ the unit square $[0, 1] \times [0, 1]$, be given with interior denoted by \mathbb{I}^2 , and the closure denoted by $\bar{\mathbb{I}}^2$. By \square_N we shall denote a partition of $\bar{\mathbb{I}}^2$ whose number of elements has order N as $N \rightarrow \infty$, and is so that the majority of elements are squares, however some small number ($o(N)$ as $N \rightarrow \infty$) of rectangles is also allowed. The interior of an arbitrary element of the partition \square_N we shall denote by Ω^N , the closure by $\bar{\Omega}^N$, and the boundary by $\partial\Omega^N$.

We shall need the following standard notation: $C(\Omega)$ denotes the space of continuous in some region Ω functions, $C^2(\Omega)$ denotes the space of twice differentiable in Ω functions, and $L_p(\Omega)$, $1 \leq p < \infty$ denotes the space of measurable and integrable in power p functions $f : \Omega \rightarrow \mathbb{R}$ with the norm

$$\|f\|_{L_p(\Omega)} = \left(\int_{\Omega} |f(\mathbf{x})|^p d\mathbf{x} \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

In order to introduce the set of functions that we will use as an approximation tool, let us denote by $H(\Omega)$ the set of harmonic functions in bounded domain $\Omega \subset \mathbb{R}^2$, i.e.

$$H(\Omega) = \{u(\mathbf{x}) \in C^2(\Omega) : \Delta u(\mathbf{x}) = 0\},$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplace operator.

Now we shall consider the set $S(\square_N)$ of continuous on $\bar{\mathbb{I}}^2$ functions such that when restricted to any Ω^N from the partition \square_N are harmonic functions, i.e.

$$S(\square_N) = \{g \in C(\bar{\mathbb{I}}^2) : g|_{\Omega^N} = u_i|_{\Omega^N}, u_i \in H(\Omega^N) \forall \Omega^N \in \square_N\}.$$

From now on we shall refer to the functions from set $S(\square_N)$ as *harmonic splines*.

Let now a function $f \in C^2(\bar{\mathbb{I}}^2)$ is given. We will approximate it by a harmonic spline $s(f, \square_N)$ in such a way that their values coincide on the boundary of elements Ω^N . In other words we shall require the spline $s(f, \square_N)$ to "interlinate" the given function on the boundary of all partition elements. Hence, the function $s(f, \square_N)$ in each Ω^N shall satisfy the equation

$$\Delta s(f, \square_N) = 0$$

and the boundary condition

$$s(f, \square_N)|_{\partial\Omega^N} = f|_{\partial\Omega^N}.$$

In other words, function $s(f, \square_N)$ in each domain \square_N must be a solution of Dirichlet problem for Laplace equation, which is unique (see, for example, [13]) and therefore harmonic spline $s(f, \square_N)$ for each $f \in C^2(\bar{\mathbb{I}}^2)$ is well defined on $\bar{\mathbb{I}}^2$.

The main goal of this paper is to study the minimal L_p -error ($1 \leq p < \infty$) of approximation of a given function f by harmonic splines $s(f, \square_N)$ for all possible partitions \square_N which we shall denote by

$$R_N(f, L_p) = \inf_{\square_N} \|f - s(f, \square_N)\|_{L_p(\mathbb{I}^2)}.$$

The value of $R_N(f, L_p)$ we shall call the *optimal error*. The partition $\tilde{\square}_N$ on which the optimal error is achieved will be called *optimal partition*, i.e.

$$R_N(f, L_p) = \|f - s(f, \tilde{\square}_N)\|_{L_p(\mathbb{I}^2)}.$$

For an arbitrary function f it is impossible (except for some trivial cases) to find the optimal partition $\tilde{\square}_N$ and explicitly compute the value of the optimal error. Therefore, we are interested in the following two natural questions: first of all, how the optimal error behaves as number of partition elements increases ($N \rightarrow \infty$), and secondly, how to construct a sequence of partitions $\{\square_N^*\}_{N=1}^\infty$, that is *asymptotically optimal*, i.e.

$$\lim_{N \rightarrow \infty} \frac{\|f - s(f, \square_N^*)\|_{L_p(\mathbb{I}^2)}}{R_N(f, L_p)} = 1.$$

In order to state the main result of this paper, we need the following notation. By $G_\Omega(\mathbf{x}; \mathbf{v})$ with $\mathbf{x}, \mathbf{v} \in \Omega$ we shall denote the Green's function of the Dirichlet problem for the domain Ω (for the detailed definition, see, for example, [13]). In addition, denote by

$$I(\mathbf{x}) := \int_{\mathbb{I}^2} G_{\mathbb{I}^2}(\mathbf{x}; \mathbf{v}) d\mathbf{v}, \quad \mathbf{x} \in \mathbb{I}^2. \quad (2.1)$$

The main result of this paper is the following theorem.

Theorem 2.1 *For an arbitrary function $f \in C^2(\mathbb{I}^2)$ there exists a sequence $\{\square_N^*\}_{N=1}^\infty$ and corresponding sequence of splines $\{s(f, \square_N^*)\}_{N=1}^\infty$, such that*

$$\lim_{N \rightarrow \infty} N \|f - s(f, \square_N^*)\|_{L_p(\mathbb{I}^2)} = \|I\|_{L_p(\mathbb{I}^2)} \|\Delta f\|_{L_{\frac{p}{p+1}}(\mathbb{I}^2)}.$$

Remark 1. Note that the majority of the partition elements from \square_N^* are squares. This is rather restrictive condition on the mesh. However, it is somewhat natural to expect since we are using more information about the function than in the case of interpolation by polynomial splines.

Remark 2. In the case when the domain of the function is rectangle, the mesh elements must have the same shape as the domain, i.e. the same width to length ratio.

Comparing this result with analogous results in the case of approximating the given function from $C^2(\mathbb{I}^2)$ by bilinear polynomial splines ([3, 5]), we see that the order of the error of approximation is the same. Therefore, depending on the particular problem one could choose either polynomial or harmonic splines, the latter providing interpolation on a larger set, but allowing less flexible partitions.

3 Idea of the proof and auxiliary results

The main idea of the construction of each partition in the sequence is to use first “intermediate approximation” of the given function f which needs to be approximated by splines. As an intermediate approximation we shall use the piecewise function “glued” from second degree Taylor polynomials of f . Then the intermediate approximation is used to build the asymptotically optimal sequence of partitions of the domain and, consequently, the sequence of corresponding harmonic splines.

The process of constructing this sequence of partitions consists of two steps:

1. The domain $\bar{\mathbb{I}}^2$ is first divided into some number (small comparing to N) of equal sizesubdomains, and on each instead of the original function f we consider its second degree Taylor polynomial constructed at, say, center of the corresponding subdomain. The number of these subdomains is chosen so that the absolute value of the difference between the function and its Taylor polynomial is small enough on each corresponding subdomain.
2. Next, each subdomain is refined further, depending on the Taylor polynomial (and therefore the original function) on this subdomain. The final partition \square_N^* will consist of the squares and possibly some rectangles (their number is small comparing to N). The total number of partition elements is determined from the following two conditions: first of all, the total number of all the elements in \square_N^* is approximately equal to N , and secondly, the global error of approximation of piecewise quadratic function (consisting of Taylor polynomials built on the previous step), by corresponding harmonic splines is minimal.

The problem of finding this error of approximation on “smallest’ subdomain elements is reduced to approximating the function $Q(\mathbf{x}) = Ax_1^2 + Bx_2^2$, where A and B are constants, since the rest of the terms in second degree Taylor polynomial constitute a harmonic function.

The constructed in such a way partition \square_N^* for each fixed N is used in the proof of the main Theorem 2.1.

The idea of this construction based on intermediate approximation by piecewise quadratic functions has been already used in papers [1], [2], [3], [4], [5] by authors and co-authors to build asymptotically optimal sequences of polynomial splines on triangulations, rectangular partitions, and their generalizations in various settings. More on the history of adaptive approximation and asymptotically optimal sequences of splines in the case of polynomial splines can be found in [2], [3].

Let us turn now to the auxiliary results that we shall need to prove the main theorem of this paper.

For a function $f \in C^2(\bar{\mathbb{I}}^2)$ we shall define a modulus of continuity as follows

$$\omega(f, \delta) = \sup\{|f(\mathbf{x}) - f(\mathbf{x}')| : |x_1 - x'_1| \leq \delta, |x_2 - x'_2| \leq \delta, \mathbf{x}, \mathbf{x}' \in \bar{\mathbb{I}}^2\}.$$

Then we consider

$$\omega(\delta) = \max\{\omega(f_{xx}, \delta), \omega(f_{yy}, \delta), \omega(f_{xy}, \delta)\}. \quad (3.1)$$

Lemma 3.1 *Let function $f \in C^2(\mathbb{I}^2)$ be given and let $P_2(\mathbf{x}) = P_2(f; \mathbf{x}; \mathbf{x}_0)$ be its second order Taylor polynomial taken at the point \mathbf{x}_0 , that is the center of some square $D_h \subset \mathbb{I}^2$ with side length h . Then*

$$|f(\mathbf{x}) - P_2(\mathbf{x})| \leq \frac{h^2}{2} \omega\left(\frac{h}{2}\right), \quad \mathbf{x} \in D_h. \quad (3.2)$$

For the proof of this lemma see, for example, [1].

To state the next lemma we shall need the following notation.

Let $G_\Omega(\mathbf{x}; \mathbf{v})$ be a Green's function of the interior Dirichlet problem for bounded region $\Omega \subset \mathbb{R}^2$. By $\Omega + \mathbf{d}$ we denote the region obtained by the shift of the region Ω by a vector $\mathbf{d} \in \mathbb{R}^2$, and by $\alpha\Omega$ we denote the region obtained by scaling Ω by coefficient α .

Lemma 3.2 *The following properties of the Green's function hold true:*

$$G_{\alpha\Omega}(\mathbf{x}; \mathbf{v}) = G_\Omega\left(\frac{\mathbf{x}}{\alpha}; \frac{\mathbf{v}}{\alpha}\right), \quad \mathbf{x} \in \alpha\Omega, \mathbf{v} \in \overline{\alpha\Omega}, \quad (3.3)$$

$$G_{\Omega+\mathbf{d}}(\mathbf{x}; \mathbf{v}) = G_\Omega(\mathbf{x} - \mathbf{d}; \mathbf{v} - \mathbf{d}), \quad \mathbf{x} \in \Omega + \mathbf{d}, \mathbf{v} \in \overline{\Omega + \mathbf{d}}. \quad (3.4)$$

Proof. In the proof of this lemma we shall use the well-known (see, for instance, [13]) formula for the Green's function of the interior Dirichlet problem for the bounded region Ω . For $\mathbf{x} \in \Omega$, $\mathbf{v} \in \overline{\Omega}$ the Green's function can be written as

$$G_\Omega(\mathbf{x}; \mathbf{v}) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{v}|} + g(\mathbf{x}; \mathbf{v}),$$

where function $g(\mathbf{x}; \mathbf{v})$ is harmonic in both arguments on Ω , continuous in \mathbf{v} on $\overline{\Omega}$, and is chosen so that the Green's function has zero value on the boundary.

For $\mathbf{x} \in \alpha\Omega$, $\mathbf{v} \in \overline{\alpha\Omega}$

$$\begin{aligned} G_\Omega\left(\frac{\mathbf{x}}{\alpha}; \frac{\mathbf{v}}{\alpha}\right) &= \frac{1}{2\pi} \ln \frac{1}{\left|\frac{\mathbf{x}}{\alpha} - \frac{\mathbf{v}}{\alpha}\right|} + g\left(\frac{\mathbf{x}}{\alpha}; \frac{\mathbf{v}}{\alpha}\right) = \frac{1}{2\pi} \ln \frac{\alpha}{|\mathbf{x} - \mathbf{v}|} + g\left(\frac{\mathbf{x}}{\alpha}; \frac{\mathbf{v}}{\alpha}\right) = \\ &= \frac{1}{2\pi} \left(\ln \alpha + \ln \frac{1}{|\mathbf{x} - \mathbf{v}|} \right) + g\left(\frac{\mathbf{x}}{\alpha}; \frac{\mathbf{v}}{\alpha}\right) = \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{v}|} + \tilde{g}(\mathbf{x}; \mathbf{v}), \end{aligned}$$

where function $\tilde{g}(\mathbf{x}; \mathbf{v}) = g\left(\frac{\mathbf{x}}{\alpha}; \frac{\mathbf{v}}{\alpha}\right) + \frac{1}{2\pi} \ln \alpha$ is clearly harmonic in both arguments in $\alpha\Omega$ and continuous in \mathbf{v} on $\overline{\alpha\Omega}$. Besides that, since the boundary of the region Ω is mapped to the boundary of the region $\alpha\Omega$, and the Green's function has zero value on $\partial\Omega$, then the expression above will be zero on $\partial(\alpha\Omega)$. Hence, (3.3) is proved.

Let us consider next the shift of the region Ω . For $\mathbf{x} \in \Omega + \mathbf{d}$, $\mathbf{v} \in \overline{\Omega + \mathbf{d}}$

$$\begin{aligned} G_\Omega(\mathbf{x} - \mathbf{d}; \mathbf{v} - \mathbf{d}) &= \frac{1}{2\pi} \ln \frac{1}{|(\mathbf{x} - \mathbf{d}) - (\mathbf{v} - \mathbf{d})|} + g(\mathbf{x} - \mathbf{d}; \mathbf{v} - \mathbf{d}) = \\ &= \frac{1}{2\pi} \ln \frac{1}{|\mathbf{x} - \mathbf{v}|} + \tilde{g}(\mathbf{x}; \mathbf{v}). \end{aligned}$$

Since all the necessary conditions for the function $\tilde{g}(\mathbf{x}; \mathbf{v}) = g(\mathbf{x} - \mathbf{d}; \mathbf{v} - \mathbf{d})$ are satisfied, then the obtained expression is the Green's function for the region $\Omega + \mathbf{d}$, and hence (3.4) is proved. \square

Next we shall consider a square region $\Omega \subset \mathbb{R}^2$ with sides parallel to the coordinate axis, and on which the following functions is defined

$$Q(\mathbf{x}) = Ax_1^2 + Bx_2^2, \quad (3.5)$$

where A and B are constants.

We shall find the L_p -error of approximation of the function $Q(\mathbf{x})$ by a harmonic function $u(Q; \mathbf{x})$ on Ω such that the values of $u(Q; \mathbf{x})$ coincide with the function $Q(\mathbf{x})$ on the boundary of Ω . In other words, for the function $u(Q; \mathbf{x})$ the following conditions must be satisfied

$$\Delta u(Q; \mathbf{x}) = 0, \quad \mathbf{x} \in \Omega \quad (3.6)$$

and

$$u(Q; \mathbf{x})|_{\partial\Omega} = Q(\mathbf{x})|_{\partial\Omega}. \quad (3.7)$$

Lemma 3.3 *For the quadratic function $Q(\mathbf{x}) = Ax_1^2 + Bx_2^2$ such that $\Delta Q(\mathbf{x}) = 2(A + B) \neq 0$, we have*

$$\|Q - u(Q)\|_{L_p(\Omega)} = 2|A + B| \cdot |\Omega|^{1+\frac{1}{p}} \|I\|_{L_p(\mathbb{I}^2)}, \quad (3.8)$$

where $|\Omega|$ denotes the area of Ω , and I is defined in (2.1).

Proof. We shall use the following representation of an arbitrary function $u(\mathbf{x}) \in C^2(\Omega) \cap C(\overline{\Omega})$ on the region $\Omega \subset \mathbb{R}^2$ with smooth enough boundary:

$$u(\mathbf{x}) = - \int_{\partial\Omega} \frac{\partial G_\Omega(\mathbf{x}; \mathbf{v})}{\partial \bar{n}} u(\mathbf{v}) d\mathbf{v} - \int_{\Omega} G_\Omega(\mathbf{x}; \mathbf{v}) \Delta u(\mathbf{v}) d\mathbf{v}, \quad \mathbf{x} \in \Omega, \quad (3.9)$$

where $\frac{\partial G_\Omega(\mathbf{x}; \mathbf{v})}{\partial \bar{n}}$ is its partial derivative in the outer normal direction to $\partial\Omega$ (see, for example, [13]).

Then for the function $Q(\mathbf{x})$, taking into account (3.9) and the fact that $\Delta Q(\mathbf{x}) = 2(A + B)$, we have

$$Q(\mathbf{x}) = - \int_{\partial\Omega} \frac{\partial G_\Omega(\mathbf{x}; \mathbf{v})}{\partial \bar{n}} Q(\mathbf{v}) d\mathbf{v} - 2(A + B) \int_{\Omega} G_\Omega(\mathbf{x}; \mathbf{v}) d\mathbf{v}, \quad \mathbf{x} \in \Omega. \quad (3.10)$$

In addition, with the help of (3.9), the solution of the problem (3.6) – (3.7) can be written as

$$u(Q; \mathbf{x}) = - \int_{\partial\Omega} \frac{\partial G_\Omega(\mathbf{x}; \mathbf{v})}{\partial \bar{n}} Q(\mathbf{v}) d\mathbf{v}, \quad \mathbf{x} \in \Omega. \quad (3.11)$$

From (3.10), (3.11), and the fact that the Green's function is nonnegative, it follows that

$$|Q(\mathbf{x}) - u(Q; \mathbf{x})| = 2|A + B| \int_{\Omega} G_\Omega(\mathbf{x}; \mathbf{v}) d\mathbf{v}, \quad \mathbf{x} \in \Omega,$$

and, hence,

$$\|Q - u(Q)\|_{L_p(\Omega)} = 2|A + B| \left(\int_{\Omega} \left(\int_{\Omega} G_{\Omega}(\mathbf{x}; \mathbf{v}) d\mathbf{v} \right)^p d\mathbf{x} \right)^{\frac{1}{p}}. \quad (3.12)$$

Our next goal is to transform the integral in the right-hand side of (3.12) so that it explicitly depends on the area of Ω . In order to do so we need the result of Lemma 3.2.

Let $\tilde{\mathbb{I}}^2$ be the square unit region such that when centered at one of its vertices and stretched by α it becomes Ω . Then the volumes of these regions are related by $|\Omega| = \alpha^2 |\tilde{\mathbb{I}}^2|$, from where we have $\alpha = \sqrt{|\Omega|}$.

Therefore, we have the following equality

$$\left(\int_{\Omega} \left(\int_{\Omega} G_{\Omega}(\mathbf{x}; \mathbf{v}) d\mathbf{v} \right)^p d\mathbf{x} \right)^{\frac{1}{p}} = \left(\int_{\alpha\tilde{\mathbb{I}}^2} \left(\int_{\alpha\tilde{\mathbb{I}}^2} G_{\tilde{\mathbb{I}}^2} \left(\frac{\mathbf{x}}{\alpha}; \frac{\mathbf{v}}{\alpha} \right) d\mathbf{v} \right)^p d\mathbf{x} \right)^{\frac{1}{p}}.$$

Changing variables we arrive at

$$\left(\int_{\Omega} \left(\int_{\Omega} G_{\Omega}(\mathbf{x}; \mathbf{v}) d\mathbf{v} \right)^p d\mathbf{x} \right)^{\frac{1}{p}} = \alpha^{2+\frac{2}{p}} \left(\int_{\tilde{\mathbb{I}}^2} \left(\int_{\tilde{\mathbb{I}}^2} G_{\tilde{\mathbb{I}}^2}(\mathbf{x}; \mathbf{v}) d\mathbf{v} \right)^p d\mathbf{x} \right)^{\frac{1}{p}}. \quad (3.13)$$

It is left to notice that due to property (3.4) of the Green's function, the last equality will not change after the shift of the region $\tilde{\mathbb{I}}^2$ by an arbitrary vector from \mathbb{R}^2 . Therefore, as $\tilde{\mathbb{I}}^2$ we may take an arbitrary rectangular region of unit area with sides parallel to the coordinate axis, in particular, we may take \mathbb{I}^2 .

Taking into account (3.13), notation (2.1), and having recalled that for the given region Ω the value $\alpha = \sqrt{|\Omega|}$, equality (3.12) becomes

$$\|Q - u(Q)\|_{L_p(\Omega)} = 2|A + B| \cdot |\Omega|^{1+\frac{1}{p}} \|I\|_{L_p(\mathbb{I}^2)}.$$

□

Now let us turn to the proof of Theorem 2.1.

4 Proof of the main result

We shall begin the proof by constructing (for a given function $f \in C^2(\tilde{\mathbb{I}}^2)$ and fixed number N) the partition \square_N^* of $\tilde{\mathbb{I}}^2$ such that the sequence $\{\square_N^*\}_{N=1}^{\infty}$ is asymptotically optimal. The idea of this construction is given in Section 3.

First of all, we determine the number of the elements of the intermediate partition. To this end, for an arbitrary fixed $\varepsilon \in (0, 1)$ we set

$$m_N := \min \left\{ m \in \mathbb{N} : \frac{1}{2} \left(\frac{1}{m} \right)^2 \omega \left(\frac{1}{2m} \right) \leq \frac{\varepsilon}{N} \right\},$$

where $\omega(\delta)$ was defined in (3.1). It is clear that $m_N \rightarrow \infty$ as $N \rightarrow \infty$.

We shall take m_N^2 as the number of elements of intermediate partition, and we shall subdivide the square \mathbb{I}^2 into equal squares D_N^l , $l = 1, \dots, m_N^2$ with the side length $\frac{1}{m_N}$. Let us show that the number of elements in intermediate partition is indeed small comparing to the total number of elements N , i.e.

$$m_N^2 = o(N), \quad N \rightarrow \infty. \quad (4.1)$$

In order to do so let us consider $\frac{N}{m_N^2}$:

$$\begin{aligned} \frac{N}{m_N^2} &= \frac{N}{2} \frac{1}{(m_N - 1)^2} \omega\left(\frac{1}{2(m_N - 1)}\right) \frac{2(m_N - 1)^2}{m_N^2 \omega\left(\frac{1}{2(m_N - 1)}\right)} \\ &\geq \varepsilon \frac{2(m_N - 1)^2}{m_N^2 \omega\left(\frac{1}{2(m_N - 1)}\right)}. \end{aligned} \quad (4.2)$$

Taking into account $\left(\frac{m_N - 1}{m_N}\right) \rightarrow 1$ and $\omega\left(\frac{1}{2m_N}\right) \rightarrow 0$ as $N \rightarrow \infty$ we obtain $\frac{N}{m_N^2} \rightarrow \infty$ as $N \rightarrow \infty$, and hence (4.1) holds true.

In order to obtain the intermediate estimates we define the functions $f_N(\mathbf{x})$ and $Q_N(\mathbf{x})$, using the notation $P_2(f; \mathbf{x}; \mathbf{x}_l)$ for Taylor polynomial of second degree for $f(\mathbf{x})$ at the center \mathbf{x}_l of each square D_l^N , as follows:

1. For $\mathbf{x} \in D_1^N$ set

$$f_N(\mathbf{x}) = P_2(f; \mathbf{x}; \mathbf{x}_1)$$

and

$$Q_N(\mathbf{x}) = \frac{1}{2} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{x}_1) x_1^2 + \frac{1}{2} \frac{\partial^2 f}{\partial x_2^2}(\mathbf{x}_1) x_2^2.$$

2. When $1 < l \leq m_N^2$ for $\mathbf{x} \in D_l^N \setminus \cup_{i=1}^{l-1} \partial D_i^N$ we set

$$f_N(\mathbf{x}) = P_2(f; \mathbf{x}; \mathbf{x}_l)$$

and

$$Q_N(\mathbf{x}) = \frac{1}{2} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{x}_l) x_1^2 + \frac{1}{2} \frac{\partial^2 f}{\partial x_2^2}(\mathbf{x}_l) x_2^2.$$

Next we shall find the number of elements to additionally subdivide each D_N^l , $l = 1, \dots, m_N^2$. For that we shall cover the square D_N^l by a ‘‘mesh’’ consisting of squares of the same fixed area, fixing, for convenience, one of the mesh vertices with one of the vertices of the square D_N^l . The intersection of D_N^l with this mesh will provide the necessary subdivision of the square. It will consist of the squares of the original mesh as well as, possibly, some rectangles along the boundary of D_N^l .

We shall consider two cases: when the partition consists only of squares and when the partition contains some rectangles along the boundary.

First, let us assume that partition of D_N^l consists of squares only. We shall find the area of the squares next. The number of the squares in the partition of D_N^l we shall

denote by \tilde{n}_l^2 . The elements itself we shall denote by $\Omega_N^{l,i}$, $i = 1, \dots, \tilde{n}_l^2$, $l = 1, \dots, m_N^2$.

Then $\bigcup_{l=1}^{m_N^2} \bigcup_{i=1}^{\tilde{n}_l^2} \Omega_N^{l,i}$ gives the partition of $\bar{\mathbb{I}}^2$ (for fixed N), and hence $\sum_{l=1}^{m_N^2} \tilde{n}_l^2 = N$.

To determine \tilde{n}_l^2 for each D_N^l we shall minimize the value of the error of approximation of $f_N(\mathbf{x})$ by the corresponding harmonic spline $s(f_N, \square_N^*)$ on $\bar{\mathbb{I}}^2$. The needed error can be written in terms of the errors on each element of the partition as

$$\|f_N - s(f_N, \square_N^*)\|_{L_p(\mathbb{I}^2)}^p = \sum_{l=1}^{m_N^2} \sum_{i=1}^{\tilde{n}_l^2} \|f_N - s(f_N, \square_N^*)\|_{L_p(\Omega_N^{l,i})}^p. \quad (4.3)$$

Therefore, we need to find the error of approximation of $f_N(\mathbf{x})$ on each element $\Omega_N^{l,i}$. Since on $\Omega_N^{l,i}$ the difference $f_N(\mathbf{x}) - Q_N(\mathbf{x})$ is a harmonic function, using Lemma 3.3, we have

$$\begin{aligned} \|f_N - s(f_N, \square_N^*)\|_{L_p(\Omega_N^{l,i})}^p &= \|Q_N - s(f_N, \square_N^*)\|_{L_p(\Omega_N^{l,i})}^p \\ &= \left| \frac{\partial^2 f}{\partial x^2}(\mathbf{x}_l) + \frac{\partial^2 f}{\partial y^2}(\mathbf{x}_l) \right|^p |\Omega_N^{l,i}|^{p+1} \|I\|_{L_p(\mathbb{I}^2)}^p = M_{f,N}^p(\mathbf{x}_l) \frac{1}{(m_N \tilde{n}_l)^{2(p+1)}} \|I\|_{L_p(\mathbb{I}^2)}^p, \end{aligned} \quad (4.4)$$

where we used the notation

$$M_{f,N}(\mathbf{x}_l) = \left| \frac{\partial^2 f}{\partial x^2}(\mathbf{x}_l) + \frac{\partial^2 f}{\partial y^2}(\mathbf{x}_l) \right|. \quad (4.5)$$

Taking into account (4.4), the equality (4.3) becomes

$$\|f_N - s(f_N, \square_N^*)\|_{L_p(\mathbb{I}^2)}^p = \sum_{l=1}^{m_N^2} M_{f,N}^p(\mathbf{x}_l) \frac{1}{(m_N \tilde{n}_l)^{2(p+1)}} \|I\|_{L_p(\mathbb{I}^2)}^p \tilde{n}_l^2,$$

from where we finally arrive at

$$\|f_N - s(f_N, \square_N^*)\|_{L_p(\mathbb{I}^2)}^p = \sum_{l=1}^{m_N^2} M_{f,N}^p(\mathbf{x}_l) \frac{1}{(m_N)^{2(p+1)} (\tilde{n}_l)^{2p}} \|I\|_{L_p(\mathbb{I}^2)}^p. \quad (4.6)$$

Using the method of Lagrange multipliers, we shall minimize the right-hand side of (4.6) under constraint

$$\sum_{l=1}^{m_N^2} \tilde{n}_l^2 = N. \quad (4.7)$$

For that let us consider the function

$$\Phi(\tilde{n}_1, \dots, \tilde{n}_{m_N^2}, \lambda) = \sum_{l=1}^{m_N^2} M_{f,N}^p(\mathbf{x}_l) \frac{1}{(m_N)^{2(p+1)} (\tilde{n}_l)^{2p}} \|I\|_{L_p(\mathbb{I}^2)}^p + \lambda \sum_{l=1}^{m_N^2} \tilde{n}_l^2, \quad (4.8)$$

where λ is the Lagrange multiplier.

To find the critical points of the function $\Phi(\tilde{n}_1, \dots, \tilde{n}_{m_N^2}, \lambda)$ we shall consider the system

$$\begin{cases} \frac{\partial}{\partial \tilde{n}_l} \left(\sum_{i=1}^{m_N^2} M_{f,N}^p(\mathbf{x}_i) \frac{1}{(m_N)^{2(p+1)} (\tilde{n}_l)^{2p}} \|I\|_{L_p(\mathbb{I}^2)}^p + \lambda \sum_{i=1}^{m_N^2} \tilde{n}_i^2 \right) = 0, \quad l = 1, \dots, m_N^2, \\ \sum_{i=1}^{m_N^2} \tilde{n}_i^2 = N. \end{cases} \quad (4.9)$$

Solving it we obtain \tilde{n}_l^2 :

$$\tilde{n}_l^2 = \frac{NM_{f,N}^{\frac{p}{p+1}}(\mathbf{x}_l)}{\sum_{i=1}^{m_N^2} M_{f,N}^{\frac{p}{p+1}}(\mathbf{x}_i)}, \quad l = 1, \dots, m_N^2. \quad (4.10)$$

Since for the solutions of the system (4.9) the value $d^2\Phi > 0$, then (4.10) indeed provide the minimal value of the total error of approximation of the function $f_N(\mathbf{x})$ by harmonic spline $s(f_N, \square_N^*)$ on the square $\bar{\mathbb{I}}^2$.

As the area of every square of the mesh $\Omega \subset D_N^l$ for the fixed N we shall take

$$|\Omega| = \frac{1}{(m_N \tilde{n}_l)^2}, \quad l = 1, \dots, m_N^2. \quad (4.11)$$

Later we shall need the following estimate for the values \tilde{n}_l

$$\tilde{n}_l \geq \frac{\sqrt{N} \min_{\mathbf{x} \in \mathbb{I}^2} \{M_{f,N}^{\frac{p}{2(p+1)}}(\mathbf{x})\}}{m_N \|\Delta f\|_{L_\infty(\mathbb{I}^2)}^{\frac{p}{2(p+1)}}} = C \frac{\sqrt{N}}{m_N} > 0, \quad (4.12)$$

where C is independent of N constant.

For the final partition of the domain to consist of only squares, the values defined by (4.10) for all $l = 1, \dots, m_N^2$ must be squares of natural numbers. In this case we will have optimal partition. Otherwise, the final partition will contain both squares and rectangles, and will be optimal only asymptotically.

From now on we shall assume that the partition of D_N^l consists of both squares of the mesh and some rectangles along the boundary. By R_N^l we shall denote the set of all squares, and by \tilde{R}_N^l - the set of rectangles from the partition of D_N^l . In addition, let $n_l^2 = n_l^2(N)$ denote the number of squares from R_N^l . Then the number of rectangles from \tilde{R}_N^l is clearly equal to $2n_l + 1$.

It is also clear that for all $l = 1, \dots, m_N^2$ we have

$$(\tilde{n}_l - 1)^2 < n_l^2 \leq \tilde{n}_l^2. \quad (4.13)$$

The number of elements of the partition of D_N^l (in the case when rectangles are present) is $(n_l + 1)^2$. Let us show that the total number of the partition elements

$\sum_{l=1}^{m_N^2} (n_l + 1)^2$ and N are values of the same order as $N \rightarrow \infty$. To that end, let us estimate

the values $\sum_{l=1}^{m_N^2} n_l^2$ and $\sum_{l=1}^{m_N^2} n_l$. Using inequality (4.13) and condition (4.7) we have

$$\sum_{l=1}^{m_N^2} n_l^2 \leq \sum_{l=1}^{m_N^2} \tilde{n}_l^2 = N, \quad (4.14)$$

$$\sum_{l=1}^{m_N^2} n_l^2 \geq \sum_{l=1}^{m_N^2} \tilde{n}_l^2 - 2 \sum_{l=1}^{m_N^2} \tilde{n}_l = N - 2 \sum_{l=1}^{m_N^2} \tilde{n}_l. \quad (4.15)$$

Using Holder inequality, we obtain

$$\sum_{l=1}^{m_N^2} n_l \leq \sum_{l=1}^{m_N^2} \tilde{n}_l \leq m_N \left(\sum_{l=1}^{m_N^2} \tilde{n}_l^2 \right)^{\frac{1}{2}} = m_N \sqrt{N}. \quad (4.16)$$

Taking into account estimates (4.14) – (4.16), we have the following double inequality

$$N - 2m_N \sqrt{N} \leq \sum_{l=1}^{m_N^2} (n_l + 1)^2 \leq N + 2m_N \sqrt{N} + m_N^2.$$

Dividing both sides by N , taking the limit as $N \rightarrow \infty$ and using (4.1), we obtain

$$\lim_{N \rightarrow \infty} \frac{\sum_{l=1}^{m_N^2} (n_l + 1)^2}{N} = 1.$$

Therefore, the number of elements of the constructed partition has order N as $N \rightarrow \infty$.

Next we shall verify that the constructed in such a way sequence of partitions will be asymptotically optimal.

Using the triangle inequality we have

$$\begin{aligned} \|f - s(f, \square_N)\|_{L_p(\mathbb{I}^2)} &\leq \|f - f_N\|_{L_p(\mathbb{I}^2)} + \|s(f, \square_N) - s(f_N, \square_N)\|_{L_p(\mathbb{I}^2)} \\ &\quad + \|f_N - s(f_N, \square_N)\|_{L_p(\mathbb{I}^2)}. \end{aligned} \quad (4.17)$$

We shall now estimate each term on the right.

For the first and second terms we shall first obtain the estimates on an arbitrary square D_N^l of intermediate partition in terms of $\|f - f_N\|_{L_\infty(D_N^l)}$. Since $|D_N^l| = \frac{1}{m_N^2}$, then the following is true

$$\|f - f_N\|_{L_p(D_N^l)}^p \leq \frac{1}{m_N^2} \|f - f_N\|_{L_\infty(D_N^l)}^p. \quad (4.18)$$

To estimate the difference $\|s(f, \square_N^*) - s(f_N, \square_N^*)\|_{L_p(D_N^l)}$ we shall consider an arbitrary element of the partition $\Omega \subset D_N^l$. According to the definition of a harmonic

spline, the difference of functions $s(f, \square_N^*)(\mathbf{x}) - s(f_N, \square_N^*)(\mathbf{x})$ is a harmonic function on Ω and its values on $\partial\Omega$ coincide with the values of the difference $f(\mathbf{x}) - f_N(\mathbf{x})$. Therefore, using representation (3.9), we obtain

$$s(f, \square_N^*)(\mathbf{x}) - s(f_N, \square_N^*)(\mathbf{x}) = - \int_{\partial\Omega} \frac{\partial G_\Omega(\mathbf{x}; \mathbf{v})}{\partial \bar{n}} (f(\mathbf{v}) - f_N(\mathbf{v})) d\mathbf{v}, \quad \mathbf{x} \in \Omega.$$

Then

$$|s(f, \square_N^*)(\mathbf{x}) - s(f_N, \square_N^*)(\mathbf{x})| \leq \|f - f_N\|_{L^\infty(\Omega)} \int_{\partial\Omega} \frac{\partial G_\Omega(\mathbf{x}; \mathbf{v})}{\partial \bar{n}} d\mathbf{v}, \quad \mathbf{x} \in \Omega.$$

Taking into account the uniqueness of the solution (3.9) (see, for instance, [13]), we have

$$\int_{\partial\Omega} \frac{\partial G_\Omega(\mathbf{x}; \mathbf{v})}{\partial \bar{n}} d\mathbf{v} = 1,$$

and, therefore, for all $\mathbf{x} \in D_N^l$

$$|s(f, \square_N^*)(\mathbf{x}) - s(f_N, \square_N^*)(\mathbf{x})| \leq \|f - f_N\|_{L^\infty(D_N^l)},$$

which implies

$$\|s(f, \square_N^*) - s(f_N, \square_N^*)\|_{L_p(D_N^l)}^p \leq \frac{1}{m_N^2} \|f - f_N\|_{L^\infty(D_N^l)}^p. \quad (4.19)$$

Using now Lemma 3.1 and definition of m_N , we obtain

$$\|f - f_N\|_{L^\infty(D_N^l)} \leq \frac{1}{2m_N^2} \omega \left(\frac{1}{2m_N} \right) \leq \frac{\varepsilon}{N}.$$

The last inequality, together with (4.19) and (4.18), imply the needed estimates:

$$\|f - f_N\|_{L_p(\mathbb{I}^2)} \leq \frac{\varepsilon}{N}, \quad (4.20)$$

$$\|s(f, \square_N^*) - s(f_N, \square_N^*)\|_{L_p(\mathbb{I}^2)} \leq \frac{\varepsilon}{N}. \quad (4.21)$$

Next let us consider the third term in (4.17)

$$\begin{aligned} \|f_N - s(f_N, \square_N)\|_{L_p(\mathbb{I}^2)}^p &= \sum_{l=1}^{m_N^2} \|f_N - s(f_N, \square_N)\|_{L_p(D_N^l)}^p \\ &= \sum_{l=1}^{m_N^2} \left(\sum_{\Omega \in \mathcal{R}_N^l} \|f_N - s(f_N, \square_N)\|_{L_p(\Omega)}^p + \sum_{\tilde{\Omega} \in \tilde{\mathcal{R}}_N^l} \|f_N - s(f_N, \square_N)\|_{L_p(\tilde{\Omega})}^p \right). \end{aligned} \quad (4.22)$$

Since for each rectangle $\tilde{\Omega} \in \tilde{\mathcal{R}}_N^l$ there exists such an element of the mesh Ω that $\tilde{\Omega} \subset \Omega$, then

$$\|f_N - s(f_N, \square_N)\|_{L_p(\tilde{\Omega})}^p \leq \|f_N - s(f_N, \square_N)\|_{L_p(\Omega)}^p.$$

Therefore, using equality (4.4) for expression (4.22) we obtain

$$\|f_N - s(f_N, \square_N)\|_{L_p(\mathbb{I}^2)}^p \leq \sum_{l=1}^{m_N^2} M_{f,N}^p(\mathbf{x}_l) \frac{1}{(m_N \tilde{n}_l)^{2(p+1)}} \|I\|_{L_p(\mathbb{I}^2)}^p (n_l + 1)^2.$$

Taking into account the estimate (4.13) for the number of elements n_l , we have

$$\begin{aligned} \|f_N - s(f_N, \square_N)\|_{L_p(\mathbb{I}^2)}^p &\leq \frac{\|I\|_{L_p(\mathbb{I}^2)}^p}{m_N^{2(p+1)}} \sum_{l=1}^{m_N^2} M_{f,N}^p(\mathbf{x}_l) \frac{(\tilde{n}_l + 1)^2}{(\tilde{n}_l)^{2(p+1)}} \\ &= \frac{\|I\|_{L_p(\mathbb{I}^2)}^p}{m_N^{2(p+1)}} \sum_{l=1}^{m_N^2} M_{f,N}^p(\mathbf{x}_l) \frac{1}{\tilde{n}_l^{2p}} \left(1 + \frac{2}{\tilde{n}_l} + \frac{1}{\tilde{n}_l^2}\right). \end{aligned}$$

We shall estimate \tilde{n}_l^{2p} using (4.10), and for \tilde{n}_l and \tilde{n}_l^2 we shall use (4.12)

$$\begin{aligned} \|f_N - s(f_N, \square_N)\|_{L_p(\mathbb{I}^2)}^p &\leq \frac{\|I\|_{L_p(\mathbb{I}^2)}^p}{m_N^{2(p+1)}} \sum_{l=1}^{m_N^2} M_{f,N}^p(\mathbf{x}_l) \frac{\left(\sum_{i=1}^{m_N^2} M_{f,N}^{\frac{p}{p+1}}(\mathbf{x}_i)\right)^p}{N^p M_{f,N}^{\frac{p^2}{p+1}}(\mathbf{x}_l)} \left(1 + \frac{2m_N}{\sqrt{N}} + \frac{m_N^2}{N}\right) \\ &= \frac{\|I\|_{L_p(\mathbb{I}^2)}^p}{N^p} \left(\frac{1}{m_N^2} \sum_{i=1}^{m_N^2} M_{f,N}^{\frac{p}{p+1}}(\mathbf{x}_i)\right)^{p+1} \left(1 + \frac{2m_N}{\sqrt{N}} + \frac{m_N^2}{N}\right). \quad (4.23) \end{aligned}$$

Since

$$\frac{1}{m_N^2} \sum_{i=1}^{m_N^2} M_{f,N}^{\frac{p}{p+1}}(\mathbf{x}_i) \rightarrow \int_{\mathbb{I}^2} \Delta f^{\frac{p}{p+1}}(\mathbf{x}) d\mathbf{x}, \quad N \rightarrow \infty,$$

then for large enough N

$$\|f_N - s(f_N, \square_N)\|_{L_p(\mathbb{I}^2)}^p \leq \frac{\|I\|_{L_p(\mathbb{I}^2)}^p}{N^p} \left((1 + \varepsilon) \int_{\mathbb{I}^2} \Delta f^{\frac{p}{p+1}}(\mathbf{x}) d\mathbf{x} \right)^{p+1} \left(1 + \frac{2m_N}{\sqrt{N}} + \frac{m_N^2}{N}\right).$$

Going back to (4.17), and taking into account (4.20) and (4.21), we obtain

$$\|f - s(f, \square_N)\|_{L_p(\mathbb{I}^2)} \leq \frac{2\varepsilon}{N} + \frac{\|I\|_{L_p(\mathbb{I}^2)}}{N} \left((1 + \varepsilon) \int_{\mathbb{I}^2} \Delta f^{\frac{p}{p+1}}(\mathbf{x}) d\mathbf{x} \right)^{\frac{p+1}{p}} \left(1 + \frac{2m_N}{\sqrt{N}} + \frac{m_N^2}{N}\right)^{\frac{1}{p}}.$$

Hence, using (4.1), we have

$$\limsup_{N \rightarrow \infty} N \|f - s(f, \square_N)\|_{L_p(\mathbb{I}^2)} = 2\varepsilon + \|I\|_{L_p(\mathbb{I}^2)} \left((1 + \varepsilon) \int_{\mathbb{I}^2} \Delta f^{\frac{p}{p+1}}(\mathbf{x}) d\mathbf{x} \right)^{\frac{p+1}{p}}.$$

Finally, since $\varepsilon \in (0, 1)$ is arbitrary, then switching to the limit as $\varepsilon \rightarrow +0$, we obtain

$$\limsup_{N \rightarrow \infty} N \|f - s(f, \square_N)\|_{L_p(\mathbb{I}^2)} = \|I\|_{L_p(\mathbb{I}^2)} \|\Delta f\|_{L_{\frac{p}{p+1}}(\mathbb{I}^2)}.$$

In order to obtain the estimate from below we shall use again the function $f_N(\mathbf{x})$ for intermediate approximation. We shall use the triangle inequality for $\|f_N - s(f_N, \square_N^*)\|_{L_p(\mathbb{I}^2)}$ as follows

$$\begin{aligned} \|f - s(f, \square_N^*)\|_{L_p(\mathbb{I}^2)} &\geq \|f_N - s(f_N, \square_N^*)\|_{L_p(\mathbb{I}^2)} - \|f - f_N\|_{L_p(\mathbb{I}^2)} \\ &\quad - \|s(f, \square_N^*) - s(f_N, \square_N^*)\|_{L_p(\mathbb{I}^2)}. \end{aligned} \quad (4.24)$$

The estimate from below for the first term can be obtained very similarly to the way we obtained the estimate from above for this term. The result of it (for large enough N) will be

$$\|f_N - s(f_N, \square_N^*)\|_{L_p(\mathbb{I}^2)} > \frac{\|I\|_{L_p(\mathbb{I}^2)}}{N} \left((1 - \varepsilon) \int_{\mathbb{I}^2} |\Delta f(\mathbf{x})|^{\frac{p}{p+1}} d\mathbf{x} \right)^{\frac{p+1}{p}} \left(1 - \frac{2m_N}{\sqrt{N}} \right)^{\frac{1}{p}}.$$

Now we may rewrite (4.24), using for second and third terms the inequalities (4.20) and (4.21)

$$\|f - s(f, \square_N^*)\|_{L_p(\mathbb{I}^2)} > \frac{\|I\|_{L_p(\mathbb{I}^2)}}{N} \left((1 - \varepsilon) \int_{\mathbb{I}^2} |\Delta f(\mathbf{x})|^{\frac{p}{p+1}} d\mathbf{x} \right)^{\frac{p+1}{p}} \left(1 - \frac{2m_N}{\sqrt{N}} \right)^{\frac{1}{p}} - \frac{2\varepsilon}{N}.$$

Multiplying both sides by N , switching to the limit as $N \rightarrow \infty$, and taking into account (4.1)

$$\liminf_{N \rightarrow \infty} N \|f - s(f, \square_N^*)\|_{L_p(\mathbb{I}^2)} \geq \|I\|_{L_p(\mathbb{I}^2)} \left((1 - \varepsilon) \int_{\mathbb{I}^2} |\Delta f(\mathbf{x})|^{\frac{p}{p+1}} d\mathbf{x} \right)^{\frac{p+1}{p}} - 2\varepsilon.$$

Since ε is arbitrary, we arrive at

$$\liminf_{N \rightarrow \infty} N \|f - s(f, \square_N^*)\|_{L_p(\mathbb{I}^2)} \geq \|I\|_{L_p(\mathbb{I}^2)} \|\Delta f\|_{L_{\frac{p}{p+1}}(\mathbb{I}^2)}.$$

Together with the estimate from above it completes the proof of Theorem 2.1. \square

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