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Fibonacci Number of the Tadpole Graph

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Abstract

In 1982, Prodinger and Tichy defined the Fibonacci number of a graph G to be the number of independent sets of the graph G. They did so since the Fibonacci number of the path graph P_n is the Fibonacci number F_{n+2} and the Fibonacci number of the cycle graph C_n is the Lucas number L_n. The tadpole graph $T_{n,k}$ is the graph created by concatenating C_n and P_k with an edge from any vertex of C_n to a pendant of P_k for integers $n = 3$ and $k = 0$. This paper establishes formulae and identities for the Fibonacci number of the tadpole graph via algebraic and combinatorial methods.

Keywords: independent sets; Fibonacci sequence; cycles; paths
Mathematics Subject Classification : 05C69

1. Introduction

Given a graph $G = (V, E)$, a set $S \subseteq V$ is an independent set of vertices if no two vertices in S are adjacent. In our illustrations, we indicate membership in an independent set S by shading the vertices in S. Let the set of all independent sets of a graph G be denoted by $I(G)$ and let $i(G) = |I(G)|$. Note that $\emptyset \in I(G)$. The *path graph*, P_n, consists of the vertex set $V = \{1, 2, \ldots, n\}$ and the edge set $E = \{\{1,2\}, \{2,3\}, \ldots, \{n-1, n\}\}$. The *cycle graph*, C_n, is the path graph, P_n,with the additional edge $\{1, n\}$.

Table 1 shows initial Fibonacci and Lucas numbers. In 1982, Prodinger and Tichy defined the Fibonacci number of a graph G, $i(G)$, to be the number of independent sets (including the empty

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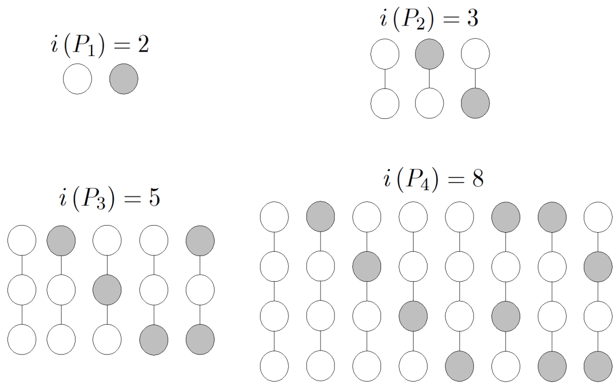

Figure 1. Independent sets of $P_1; P_2; P_3;$ and P_4.

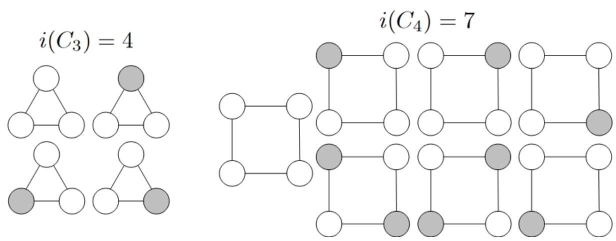

Figure 2. Independent sets of C_3 and C_4.

set) of the graph G [5]. They did so because the Fibonacci number of the path graph P_n is the Fibonacci number F_{n+2}, and the Fibonacci number of the cycle graph C_n is the Lucas number L_n.

n	0	1	2	3	4	5	6	7	8	9
F_n	0	1	1	2	3	5	8	13	21	34
L_n	2	1	3	4	7	11	18	29	47	76

Table 1: Initial values of the Fibonacci and Lucas sequences

In [1], the authors of this paper use these graphs to combinatorially derive identities relating Fibonacci and Lucas numbers.

Example 1. $L_n = F_{n-1} + F_{n+1}$ *for positive integers* $n \geq 3$.

Proof. On the one hand we know that $i(C_n) = L_n$. On the other hand, vertex 1 is either a member of the independent set or it is not. If not, then any independent set from P_{n-1}, formed by vertices 2 through n, can be selected in $i(P_{n-1})$ ways. If in the set, then the remaining members can be selected in $i(P_{n-3})$ ways from the path formed by vertices 3 through $n-1$, since vertices 2 and n can not be selected. Hence, $L_n = i(C_n) = i(P_{n-3}) + i(P_{n-1}) = F_{n-1} + F_{n+1}$. □

The Fibonacci sequence and the Lucas sequence are famous examples of the more general form called the Gibonacci sequence [3]. For integers $G_0 = a$ and $G_1 = b$, the Gibonacci sequence is defined recursively as $G_n = G_{n-1} + G_{n-2}$ for positive integers $n \geq 2$. Do other graphs exist whose Fibonacci numbers form a Gibonacci sequence?

The tadpole graph, $T_{n,k}$, is the graph created by concatenating C_n and P_k with an edge from any vertex of C_n to a pendent of P_k for integers $n \geq 3$ and $k \geq 0$. For ease of reference we label the vertices of the cycle $c_1, ..., c_n$, the vertices of the path $p_1, ..., p_k$ where c_1 is adjacent to p_1.

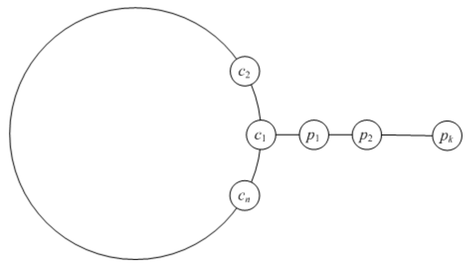

Figure 3. The Tadpole Graph $T_{n,k}$.

Example 2. *Independent sets on* $T_{3,2}$

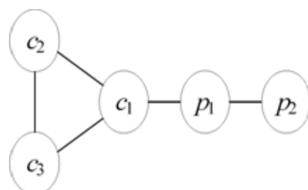

Figure 4. $T_{3,2}$.

$$I\left(T_{3,2}\right) = \{\emptyset, \{c_1\}, \{c_2\}, \{c_3\}, \{p_1\}, \{p_2\}, \{c_1, p_2\}, \{c_2, p_1\}, \{c_2, p_2\}, \{c_3, p_1\}, \{c_3, p_2\}\}$$

Theorem 1.1. $i\left(T_{n,k}\right) = i\left(T_{n,k-1}\right) + i\left(T_{n,k-2}\right).$

Proof. We show that $I\left(T_{n,k}\right) = I\left(T_{n,k-1}\right) \cup I\left(T_{n,k-2}\right)$ where $I\left(T_{n,k-1}\right) \cap I\left(T_{n,k-2}\right) = \emptyset$. Partition $I\left(T_{n,k}\right)$ into two disjoint subsets: sets where p_k is shaded and sets where p_k is not shaded. For every independent set in $I\left(T_{n,k-2}\right)$, add an unshaded vertex p_{k-1} followed by a shaded vertex p_k to the end of the path graph. For every independent set in $I\left(T_{n,k-1}\right)$, add an unshaded vertex p_k to the end of the path graph. Therefore, $i\left(T_{n,k}\right) = i\left(T_{n,k-1}\right) + i\left(T_{n,k-2}\right)$. □

Theorem 1.2. $i\left(T_{n,k}\right) = i\left(T_{n-1,k}\right) + i\left(T_{n-2,k}\right).$

Proof. Again, we show that $I\left(T_{n,k}\right) = I\left(T_{n-1,k}\right) \cup I\left(T_{n-2,k}\right)$ where $I\left(T_{n-1,k}\right) \cap I\left(T_{n-2,k}\right) = \emptyset$. Label any three consecutive vertices of $T_{n,k}$ of degree two from the cycle as $n-1$, n and 1. For every independent set in $I\left(T_{n-2,k}\right)$, if vertex 1 is shaded (then vertex $n-2$ is not shaded), insert a shaded vertex $n-1$ and an unshaded vertex n, thus creating all independent sets of $T_{n,k}$ that include both 1 and $n-1$. If vertex 1 is not shaded, then insert a shaded vertex n and unshaded vertex $n-1$ creating all independent sets where vertex n is shaded. For every independent set in $I\left(T_{n-1,k}\right)$, insert an unshaded vertex n which finally creates all independent sets where either 1 or $n-1$ is shaded or none of $n-1$, n and 1 are shaded. Therefore, $i\left(T_{n,k}\right) = i\left(T_{n-1,k}\right) + i\left(T_{n-2,k}\right)$. □

It is immediate that $i(T_{n,0}) = L_n$ since $T_{n,0} \cong C_n$. Computing $i(T_{3,2}) = 11$ and $i(T_{4,1}) = 12$ allows us to effortlessly fill in the following table since by Theorems 1.1 and 1.2, every row and column forms a Gibonacci sequence.

	k	0	1	2	3	4	5	6	7	8	9	10
n												
3		4	7	11	18	29	47	76	123	199	322	521
4		7	12	19	31	50	81	131	212	343	555	898
5		11	19	30	49	79	128	207	335	542	877	1419
6		18	31	49	80	129	209	338	547	885	1432	2317
7		29	50	79	129	208	337	545	882	1427	2309	3736
8		47	81	128	209	337	546	883	1429	2312	3741	6053
9		76	131	207	338	545	883	1428	2311	3739	6050	9789
10		123	212	335	547	882	1429	2311	3740	6051	9791	15842

Table 2: Fibonacci Numbers for the Tadpole Graph, $T_{n,k}$

It is easy to directly compute the Fibonacci number of any Tadpole graph.

Theorem 1.3. $i(T_{n,k}) = L_{n+k} + F_{n-3}F_k$.

Proof. We proceed with two base cases and strong induction on k. Suppose $k = 0$. Then $i(T_{n,0}) = i(C_n) = L_n + F_{n-3}F_0 = L_n$. For $k = 1$, combinatorially, $i(T_{n,1}) = i(C_n) + i(P_{n-1}) = L_n + F_{n+1}$. Now algebraically,

$$\begin{aligned} L_n + F_{n+1} &= L_{n+1} - L_{n-1} + F_n + F_{n-1} \\ &= L_{n+1} - F_{n-2} + F_{n-1} \\ &= L_{n+1} + F_{n-3}F_1. \end{aligned}$$

Finally, for general $k \geq 2$,

$$\begin{aligned} i(T_{n,k+1}) &= i(T_{n,k}) + i(T_{n,k-1}) \\ &= L_{n+k} + F_{n-3}F_k + L_{n+k-1} + F_{n-3}F_{k-1} \\ &= L_{n+k+1} + F_{n-3}F_{k+1}. \end{aligned}$$

□

Theorem 1.4. *For $n \geq 3$ and $k \geq 0$,*

1. $L_{n+k} = F_{n-1}F_{k+1} + F_{n+1}F_{k+2} - F_{n-3}F_k$;
2. $L_{n+k} = F_{n+1}F_k + L_nF_{k+1} - F_{n-3}F_k$;
3. $L_{n+k} = F_{n-1}F_{k+2} + F_{n+k+1} - F_{n-3}F_k$.

Proof. For 1, we know that there are $L_{n+k} + F_{n-3}F_k$ independent sets on the tadpole graph $T_{n,k}$. Now we partition $I\left(T_{n,k}\right)$ into two disjoint sets: sets that contain c_1 and sets that do not. If c_1 is included in the independent set then c_2, c_n and p_1 are not. Hence, there are $i(P_{n-3})i\left(P_{k-1}\right) = F_{n-1}F_{k+1}$ such sets. If c_1 is not included in the independent set then there are $i(P_{n-1})i\left(P_k\right) = F_{n+1}F_{k+2}$ such sets. So, $L_{n+k} + F_{n-3}F_k = F_{n-1}F_{k+1} + F_{n+1}F_{k+2}$ and the result follows.

For 2, we partition $I\left(T_{n,k}\right)$ into two disjoint sets: sets that contain p_1 and sets that do not. If p_1 is included in the independent set then c_1,and p_2 are not. Hence, there are $i(P_{n-1})i\left(P_{k-2}\right) = F_{n+1}F_k$ such sets. If p_1 is not included in the independent set then there are $i(C_n)i\left(P_{k-1}\right) = L_nF_{k+1}$ such sets. So $L_{n+k} + F_{n-3}F_k = F_{n+1}F_k + L_nF_{k+1}$ and the result follows.

For 3, we partition $I\left(T_{n,k}\right)$ into two disjoint sets: sets that contain c_n and sets that do not. If c_n is included in the independent set then c_1 and c_{n-1} are not. Hence, there are $i(P_{n-3})i\left(P_k\right) = F_{n-1}F_{k+2}$ such sets. If c_n is not included in the independent set then there are $i(P_{n-1+k}) = F_{n+k+1}$ such sets. So, $L_{n+k} + F_{n-3}F_k = F_{n-1}F_{k+2} + F_{n+k+1}$ and the result follows. □

2. Tadpole Triangle

We turn Table 2 into a triangular array where the (n, k) entry for $n \geq 3$ and $k \geq 0$ will be denoted $t_{n,k}$. Row n will represent the class of tadpole graphs with a total of n vertices. As the value of k increases by 1 through each row of the triangle, the cycle subgraph shrinks by one vertex and the length of the path subgraph increases by one. Thus, $t_{n,k}$ represents the number of independent sets on the Tadpole graph with n vertices with a path of length k (and thus, a cycle of length $n - k$). Hence, $t_{n,k} = i(T_{n-k,k})$. By Theorem 1.3, $t_{n,k} = L_n + F_{n-k-3}F_k$.

						4						
					7		7					
				11		12		11				
			18		19		19		18			
		29		31		30		31		29		
	47		50		49		49		50		47	
76		81		79		80		79		81		76

Table 3: The Triangular Array of Fibonacci Numbers of the Tadpole Graph

As noted before, $t_{n,0} = i(T_{n,0}) = L_n$. Casual observation seems to indicate the rows the tadpole triangle are symmetric.

Theorem 2.1. $t_{n,k} = t_{n,n-k-3}$

Proof. Theorem 1.3 provides a quick, algebraic proof of the symmetry of rows since $t_{n,n-k-3} = i(T_{k+3,n-k-3}) = L_n + F_kF_{n-k-3} = i(T_{n-k,k}) = t_{n,k}$. □

Proof. For a combinatorial proof of the symmetry in rows, consider c_2 in both $T_{k+3,n-k-3}$ and $T_{n-k,k}$. As before, we partition the tadpole graphs into two disjoint sets: those that contain c_2 and those that do not. Both tadpole graphs contain n vertices. Thus, the number of independent sets that do not contain c_2 in each tadpole graph is the number of independent sets on the path with

$n-1$ vertices. Independent sets that contain c_2, do not contain c_1. This decomposes the tadpole graph into two disjoint paths. For both tadpole graphs, disjoint paths of length k and $n-k-3$ are created. Both tadpole graphs lead to the same decomposition and $t_{n,k} = t_{n,n-k-3}$. □

Theorem 2.2. $t_{n,k+1} - t_{n,k} = (-1)^k F_{n-2k-4}$ *for* $0 \le k \le \left\lfloor \frac{n-3}{2} \right\rfloor$.

Proof. Algebraically,

$$\begin{aligned} t_{n,k+1} - t_{n,k} &= L_n + F_{n-k-4}F_{k+1} - (L_n + F_{n-k-3}F_k) \\ &= F_{n-k-4}F_{k+1} - F_{n-k-3}F_k \\ &= (-1)^k F_{n-2k-4} \text{ by d'Ocagne's Identity.} \end{aligned}$$

□

Proof. Combinatorially we proceed by initially considering the mapping

$$\Psi(S) = \begin{cases} S \text{ for } \{c_2, c_{n-k}\} \not\subseteq S \\ (S \setminus \{c_2\}) \cup \{c_1\} \text{ for } \{c_2, c_{n-k}\} \subseteq S \end{cases}$$

from $I(T_{n-k,k})$ to $I(T_{n-k-1,k+1})$ as illustrated in Figure 5. The identity mapping pairs together most independent sets but encounters obvious problems since independent sets in $I(T_{n-k,k})$ that contain both c_2 and c_{n-k} do not map to $I(T_{n-k-1,k+1})$, and independent sets in $I(T_{n-k-1,k+1})$ that contain both $c_1$and c_{n-k} have no pre-image in $I(T_{n-k,k})$. If c_2 and c_{n-k} are both in the independent set, then remove c_2 from the independent set while including c_1 to create an independent set in $I(T_{n-k-1,k+1})$ to upgrade the identity mapping to $\Psi(S)$. We now have two subtle issues which provide the value of $t_{n,k+1} - t_{n,k}$. Independent sets in $I(T_{n-k,k})$ that contain the subset $\{p_1, c_2, c_{n-k}\}$ have no image. There are $i(P_{n-k-5})i(P_{k-2}) = F_{n-k-3}F_k$ such sets. Independent sets in $I(T_{n-k-1,k+1})$ that contain the subset $\{c_1, c_2, c_{n-k}\}$ have no pre-image. There are $i(P_{n-k-6})i(P_{k-1}) = F_{n-k-4}F_{k+1}$ such sets. Once again, $t_{n,k+1} - t_{n,k} = F_{n-k-4}F_{k+1} - F_{n-k-3}F_k = (-1)^k F_{n-2k-4}$. □

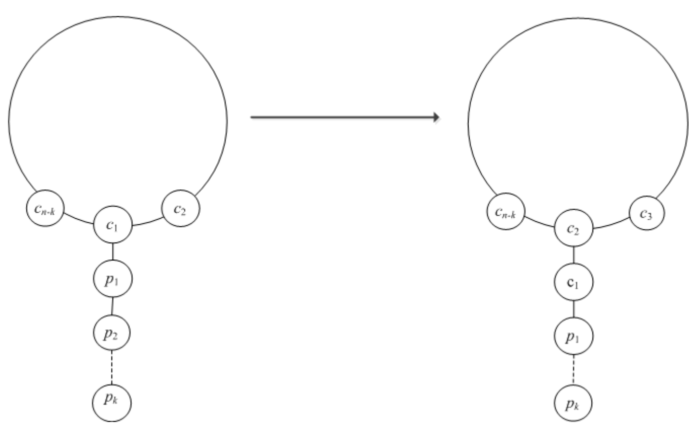

Figure 5. Mapping $I(T_{n-k,k})$ to $I(T_{n-k-1,k+1})$.

Theorem 2.3. $\sum_{k=0}^{n-3}(-1)^k t_{n,k} = \begin{cases} 0 \textit{ for even } n \\ 2F_n \textit{ for odd n.} \end{cases}$

Proof. For even n the result is trivial due to the symmetry of row values. For odd n, we proceed by induction. Base cases abound from Table 3. Assume n is odd and $\sum_{k=0}^{n-3}(-1)^k t_{n,k} = 2F_n$. Moving on to the next odd value we consider

$$\begin{aligned}
\sum_{k=0}^{n-1}(-1)^k t_{n+2,k} &= \left(\sum_{k=0}^{n-3}(-1)^k t_{n+2,k}\right) + (-1)^{n-2}t_{n+2,n-2} + (-1)^{n-1}t_{n+2,n-1} \\
&= \sum_{k=0}^{n-3}(-1)^k t_{n+2,k} - t_{n+2,n-2} + L_{n+2} \\
&= \left(\sum_{k=0}^{n-3}(-1)^k [t_{n,k} + t_{n+1,k}]\right) - L_{n+2} + F_{n+2-4} + L_{n+2} \\
&= \sum_{k=0}^{n-3}(-1)^k t_{n,k} + (-1)^k t_{n+1,k} - L_{n+2} - F_{n-2} + L_{n+2} \\
&= \left(\sum_{k=0}^{n-3}(-1)^k t_{n,k}\right) + \left(\sum_{k=0}^{n-2}(-1)^k t_{n+1,k}\right) + t_{n+1,n-2} - L_{n+2} - F_{n-2} + L_{n+2} \\
&= 2F_n + 0 + t_{n+1,n-2} - F_{n-2} = 2F_n + L_{n+1} - F_{n-2} \\
&= 3F_n + F_{n+2} - F_{n-2} = 2F_n + F_{n-1} + F_{n+2} \\
&= F_n + F_{n+1} + F_{n+2} = 2F_{n+2}
\end{aligned}$$

□

Theorem 2.4. *The ratio of consecutive row sums converges to the golden ratio ϕ.*

Proof. The sum of row n can be written as

$$\begin{aligned}
\sum_{k=0}^{n-3} t_{n,k} &= \sum_{k=0}^{n-3} (L_n + F_{n-k-3}F_k) \\
&= \sum_{k=0}^{n-3} L_n + \sum_{k=0}^{n-3} F_{n-k-3}F_k \\
&= (n-2)L_n + \frac{(n-3)L_{n-3} - F_{n-3}}{5}
\end{aligned}$$

Now,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{(n+1-2)L_{n+1} + \frac{(n+1-3)L_{n+1-3}-F_{n+1-3}}{5}}{(n-2)L_n + \frac{(n-3)L_{n-3}-F_{n-3}}{5}} \\
&= \lim_{n\to\infty} \frac{(n-1)L_{n+1} + \frac{(n-2)L_{n-2}-F_{n-2}}{5}}{(n-2)L_n + \frac{(n-3)L_{n-3}-F_{n-3}}{5}} \\
&= \lim_{n\to\infty} \frac{nL_{n+1} + nL_{n-2} - F_{n-2}}{nL_n + nL_{n-3} - F_{n-3}} \\
&= \lim_{n\to\infty} \frac{L_{n+1} + L_{n-2}}{L_n + L_{n-3}} \\
&= \lim_{n\to\infty} \frac{L_n}{L_{n-1}} = \phi.
\end{aligned}$$

□

A perfect matching (or 1-factor) in a graph $G = (V, E)$ is a subset S of edges of E such that every vertex in V is incident to exactly one edge in S. In [2], Gutman and Cyvin define the L-shaped graph, $L_{p,q}$, to be the graph with $p+q+1$ copies of C_4 as illustrated in Figure 6 by $L_{2,1}$.

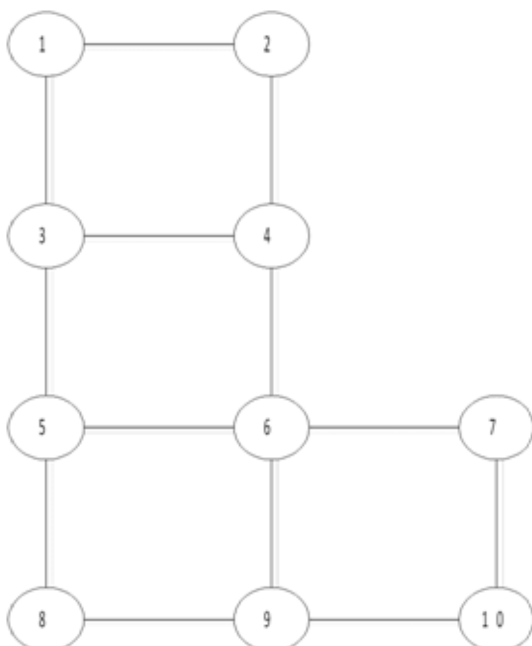

Figure 6. $L_{2,1}$.

$$\{\{1,2\},\{3,4\},\{5,6\},\{7,10\},\{8,9\}\}$$
$$\{\{1,2\},\{3,4\},\{5,8\},\{6,9\},\{7,10\}\}$$
$$\{\{1,2\},\{3,5\},\{4,6\},\{7,10\},\{8,9\}\}$$
$$\{\{1,2\},\{3,4\},\{5,8\},\{6,7\},\{9,10\}\}$$
$$\{\{1,3\},\{2,4\},\{5,6\},\{8,9\},\{7,10\}\}$$
$$\{\{1,3\},\{2,4\},\{5,8\},\{6,7\},\{9,10\}\}$$
$$\{\{1,3\},\{2,4\},\{5,8\},\{6,9\},\{7,10\}\}$$

Table 4: The seven perfect matchings of $L_{2,1}$

They show that the number of perfect matchings in $L_{p,q}$ is $F_{p+q+2} + F_{p+1}F_{q+1}$. These values correspond to the columns of the tadpole triangle. This correspondence provides a quick proof of the symmetry of rows of the tadpole triangle since $L_{p,q} \approx L_{q,p}$. We number columns starting with the center at $i = 0$.

Theorem 2.5. *The number of perfect matchings in $L_{p,q}$ is given by $t_{p+q+1,q-1}$, the p^{th} entry in columns $\pm(p-q)$.*

Proof. Since the tadpole triangle is symmetric we can assume that $p \geq q$. By Theorem 1.3,

$$\begin{aligned} t_{p+q+1,q-1} &= L_{p+q+1} + F_{p-1}F_{q-1} \\ &= F_{p+q+2} + F_{p+q} + F_{p-1}F_{q-1} \\ &= F_{p+q+2} + (F_{p+1}F_{q+1} - F_{p-1}F_{q-1}) + F_{p-1}F_{q-1} \\ &= F_{p+q+2} + F_{p+1}F_{q+1}. \end{aligned}$$

□

3. Future Work

In [4], Pederson and Vestergaard show that for every unicyclic graph G of order n, $L_n \leq i(G) \leq 3 \times 2^{n-3} + 1$. Furthermore they show that the minimum bound is realized only for $T_{n,0} \approx C_n$ and $T_{3,n-3}$. The maximum bound occurs only for C_4 and the graph with $n-3$ pendants adjacent to the same vertex of C_3. The technique of this paper can be used to precisely determine the Fibonacci number of many classes of unicyclic graphs.

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References

[1] J. DeMaio and J. Jacobson, Fibonacci and Lucas identities via graphs, *Springer Proceedings in Mathematics & Statistics: Topics from the 8th Annual UNCG Regional Mathematics and Statistics Conference*,(pp. 85–92). Springer New York.

[2] I. Gutman and S. Cyvin, A result on 1-factors related to Fibonacci numbers, *Fibonacci Quarterly* **28(1)** (1990), 81–84.

[3] T. Koshy, *Fibonacci and Lucas numbers with applications*, Wiley-Interscience, New York, 2001.

[4] A.S. Pederson and P.D. Vestergaard, The number of independent sets in unicyclic graphs, *Discrete Applied Mathematics* **152** (2005), 246–256

[5] H. Prodinger and R. Tichy, Fibonacci numbers of graphs, *Fibonacci Quarterly* **20(1)** (1982), 16–21.