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Pólya's Theorem with Zeros

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Abstract

Let $\mathbb{R}[X] := \mathbb{R}[X_1, \dots, X_n]$. Pólya's Theorem says that if a form (homogeneous polynomial) $p \in \mathbb{R}[X]$ is positive on the standard n-simplex Δ_n, then for sufficiently large N all the coefficients of $(X_1 + \cdots + X_n)^N p$ are positive. This paper is the culmination of a project to characterize forms, possibly with zeros on Δ_n, which satisfy a slightly relaxed version of Pólya's Theorem (in which the condition of "positive" is replaced by "nonnegative") and to give a bound for the N needed. In this paper we give such a characterization along with a bound. This is a broad generalization of previous results of the authors.

Key Words: Pólya's Theorem, positive polynomials, sums of squares

1 Introduction

Let $\mathbb{R}[X] := \mathbb{R}[X_1, \dots, X_n]$ and let $\mathbb{R}^+[X]$ denote polynomials in $\mathbb{R}[X]$ with nonnegative coefficients. We write Δ_n for the standard n-simplex

$$\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0, \sum x_i = 1\}.$$

Pólya's Theorem [5] says that if p is a homogeneous polynomial in n variables which is positive on Δ_n, then for a sufficiently large exponent $N \in \mathbb{N}$, all of the coefficients of $(X_1 + \cdots + X_n)^N p$ are positive. This elegant and beautiful result has many applications, both in pure and applied mathematics.

In [7], the second and third authors gave an explicit bound for the exponent N in terms of the degree, the size of the coefficients, and the minimum of p on the simplex. The current paper is the culmination of a project, begun in [6] and [8],

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to characterize forms, possibly with zeros on Δ_n, which satisfy a slightly relaxed version of Pólya's Theorem (in which the condition of "positive" is replaced by "nonnegative") and to give a bound for the N needed. In this paper we give such a characterization along with a bound. This is a broad generalization of the results in [6] and [8].

There are recent results by other authors related to the work in this paper. Recently, H.-M. Mok and W.-K. To [4] gave a sufficient condition for a form to satisfy the relaxed version of Pólya's Theorem, along with a bound in this case. In [9], S. Burgdorf, C. Scheiderer, and M. Schweighofer look at more general questions on polynomial identities certifying strict or non-strict positivity of a polynomial on a closed set in $\mathbb{R}^n$. As a corollary to one of their results, they give a sufficient condition for the relaxed Pólya's Theorem to hold for a form, involving the positivity of the partial derivatives of a form on faces of the simplex. For both of these results, the condition given is sufficient but not necessary; they can be deduced from our results.

The original Pólya's Theorem with bound from [7] has been used by other authors in applications. For example, in [10] it is used to give an algorithmic proof of Schmüdgen's Positivstellensatz, and in [2] it is used to give results on approximating the stability number of a graph. Also, in [3], an easy generalization of Pólya's Theorem and the bound to a noncommutative setting is given and used to construct relaxations for some semidefinite programming problems which arise in control theory. We believe that the results in this paper should have broad application to these and other areas.

2 Preliminaries

Let $Po(n, d)$ be the set of forms of degree d in n variables for which there exists an $N \in \mathbb{N}$ such that $(X_1 + ... + X_n)^N p \in \mathbb{R}^+[X]$. In other words, $Po(n, d)$ are the forms which satisfy the conclusion of Pólya's Theorem, with "positive coefficients" replaced by "nonnegative coefficients."

For $I \subseteq \{1, ..., n\}$, let $F(I)$ denote the face of Δ_n given by

$$\{(x_1, \ldots, x_n) \in \Delta_n \mid x_i = 0 \text{ for } i \in I\}.$$

The *relative interior* of the face $F(I)$ is the set

$$\{(x_1, \ldots, x_n) \in F(I) \mid x_j > 0 \text{ for all } j \in I^c\},$$

where I^c denotes $\{1, \ldots, n\} \setminus I$. For $f(x) \in \mathbb{R}[X]$, $Z(f)$ denotes the zeros of f.

Given $f = \sum_{\alpha \in \mathbb{N}} a_\alpha X^\alpha \in \mathbb{R}[X]$, let $\mathrm{supp}(f)$ denote $\{\alpha \in \mathbb{N} \mid a_\alpha \neq 0\}$ and define

$$\Lambda^+(f) := \{\alpha \in \mathrm{supp}(f) \mid a_\alpha > 0\}, \quad \Lambda^-(f) := \{\beta \in \mathrm{supp}(f) \mid a_\beta < 0\}.$$

For $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ in $\mathbb{N}^n$, we write $\alpha \preceq \beta$ if $\alpha_i \leq \beta_i$ for all i, and $\alpha \prec \beta$ if $\alpha \preceq \beta$ and $\alpha \neq \beta$.

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}$ and a face $F = F(I)$ of Δ_n, it will be convenient to use the notation α_F for $(\tilde{\alpha}_1, \ldots \tilde{\alpha}_n) \in \mathbb{N}$, where $\tilde{\alpha}_i = \alpha_i$ for $i \in I$ and $\tilde{\alpha}_j = 0$ for $j \in I^c$. Then $\alpha_F \preceq \beta_F$ iff $\alpha_i \leq \beta_i$ for all $i \in I$. (This is denoted $\alpha \preceq_F \beta$ in [1] and [4].)

For a form $p = \sum a_\alpha X^\alpha \in \mathbb{R}[X]$, we write $p = p^+ - p^-$ where $p^+, p^- \in \mathbb{R}^+[X]$. Then for $N \in \mathbb{N}$ and $d = \deg p$, we have

$$\begin{aligned}(X_1 + \cdots + X_n)^N p =& (X_1 + \cdots + X_n)^N (p^+ - p^-) \\ =& (X_1 + \cdots + X_n)^N p^+ - (X_1 + \cdots + X_n)^N p^- \\ =& \sum_{|\gamma| = N+d} A_\gamma X^\gamma - \sum_{|\gamma| = N+d} B_\gamma X^\gamma.\end{aligned}$$

We call A_γ the positive part and B_γ the negative part of the coefficient of X^γ. From calculations given in [7], we have

$$A_\gamma = \sum_{\substack{\alpha \in \Lambda^+(p) \\ \alpha \preceq \gamma}} \frac{N!}{(\gamma_1 - \alpha_1)! \cdots (\gamma_n - \alpha_n)!} \cdot a_\alpha \tag{1}$$

$$B_\gamma = \sum_{\substack{\beta \in \Lambda^-(p) \\ \beta \preceq \gamma}} \frac{N!}{(\gamma_1 - \beta_1)! \cdots (\gamma_n - \beta_n)!} \cdot a_\beta \tag{2}$$

We begin with some simple observations about forms in $Po(n, d)$.

Proposition 1. *Suppose $p \in Po(n, d)$.*

1. *If u a point in the relative interior of a face F of Δ_n and $p(u) = 0$, then p vanishes everywhere F. In particular, if $p(u) = 0$ for u an interior point of Δ_n, then p is the zero form.*

2. *$Z(p) \cap \Delta_n$ is a union of faces of Δ_n.*

3. *If $\beta \in \Lambda^-(p)$, then for every proper nonzero face F of Δ_n, there is $\alpha \in \Lambda^+(p)$ so that $\alpha_F \preceq \beta_F$.*

Proof. We note that (1) is easy (a proof is given in [8, §3] and [1, Prop. 2]) and (2) follows immediately from 1.

For (3), without loss of generality we can assume $F = F(\{1, \ldots, r\})$ with $1 \leq r < n$. We have $N \in \mathbb{N}$ with $(\sum X_i)^N p \in \mathbb{R}^+[X]$. Let $\gamma = (\beta_1, \ldots, \beta_{n-1}, \beta_n + N) \in \mathbb{N}^n$, then $|\gamma| = N + d$ and $\beta \preceq \gamma$. Write the coefficient of X^γ in $(\sum X_i)^N p$ as $A_\gamma - B_\gamma$ as above, then since $\beta \preceq \gamma$, by (2), $B_\gamma > 0$. Since the coefficient of X^γ in $(\sum X_i)^N p$ must be non-negative, this implies $A_\gamma > 0$ and hence, by (1), there is $\alpha \in \Lambda^+(p)$ with $\alpha \preceq \gamma$. This in turn implies $\alpha_F \preceq \beta_F$, which proves (3).

□

Remark 1. The conditions in the proposition are necessary but not sufficient conditions for $p \in Po(n,d)$. This follows from [1, Example 2] or [4, Example 5.1].

In [4], H.-N. Mok and W.-K. To give a sufficient condition for $p \in Po(n,d)$ which is related to (3) in Proposition 1.

Theorem 1 ([4], Theorem 2)**.** *Suppose p is a form of degree d such that $p \geq 0$ on Δ_n, $Z(p) \cap \Delta_n$ is a union of faces of Δ_n, and p satisfies the following property: For every face F of Δ_n with $F \subseteq Z(p)$ and each $\beta \in \Lambda^-(p)$, there is $\alpha \in \Lambda^+(p)$ such that $\alpha_F \prec \beta_F$. Then $p \in Po(n,d)$.*

This theorem will follow easily from our main theorem. The following example shows that the condition in the above theorem is not necessary.

Example. Let $p_a(X_1,X_2,X_3,X_4) = X_1^4 + X_2^4 + X_1^2(X_3^2 - aX_3X_4 + X_4^2)$, where $0 < a \leq 2$. Then $p \geq 0$ on Δ_4, $Z(p) \cap \Delta_n = F(\{1,2\})$, $\Lambda^-(p) = \{(2,0,1,1)\}$, and $\Lambda^+(p) = \{(4,0,0,0),(0,4,0,0),(2,0,0,2),(2,0,2,0)\}$. Hence all conditions of Proposition 1 hold. Note that there is no $\alpha \in \Lambda^+(p)$ with $\alpha_F \prec (2,0,1,1)_F$, where $F = F(\{1,2\})$, so that the condition of Theorem 1 doesn't hold.

If $\sum_{j=1}^4 \gamma_j = N+4$, then the coefficient of $\frac{N!}{\gamma_1!\gamma_2!\gamma_3!\gamma_4!}X_1^{\gamma_1}X_2^{\gamma_2}X_3^{\gamma_3}X_4^{\gamma_4}$ in $(\sum_j X_j)^N p_a$ is

$$\begin{aligned} f_a(\gamma_1,\gamma_2,\gamma_3,\gamma_4) := {} & \gamma_1(\gamma_1-1)(\gamma_1-2)(\gamma_1-3) + \gamma_2(\gamma_2-1)(\gamma_2-2)(\gamma_2-3) \\ & +\gamma_1(\gamma_1-1)(\gamma_3(\gamma_3-1) + \gamma_4(\gamma_4-1) - a\gamma_3\gamma_4), \end{aligned} \tag{3}$$

where any factor of the type $\gamma_i - m$ which is negative is set to 0.

We want to determine the smallest N so that for all such γ, $f_a(\gamma) \geq 0$. We first observe that

$$\begin{aligned} f_a(2,3,k,k) &= 2(2k(k-1) - ak^2) = 2k((2-a)k - 2), \\ f_a(3,3,k,k) &= 6(2k(k-1) - ak^2) = 6k((2-a)k - 2). \end{aligned}$$

If $a = 2$, then $f_a(3,3,k,k) < 0$, so no N will ever work. Otherwise, assume $0 < a < 2$ and observe that $f_a(3,3,k,k) < 0$ if

$$k < \frac{2}{2-a}.$$

Thus, if $N = 2M$ and all coefficients are non-negative, we have

$$2M + 4 = 6 + 2k \implies N = 2k + 2 \geq 2 + 2\left\lceil \frac{2}{2-a} \right\rceil$$

and if $N = 2M+1$ and all coefficients are non-negative, we have

$$2M + 5 = 6 + 2k \implies N = 2k + 1 \geq 1 + 2\left\lceil \frac{2}{2-a} \right\rceil$$

Thus the smallest N satisfies the equation

$$N \geq 1 + 2\left\lceil \frac{2}{2-a} \right\rceil.$$

A messy calculation, which we omit, shows that $p_1 \in Po(n,d)$ for all $0 < a < 2$, and there is a smallest such $N = 4(2-a)^{-1} + \mathcal{O}(1)$. In the last section, we will see that the first statement and a bound that is asymptotically the same follows from our main results.

3 Local Versions of Pólya's Theorem

For $\alpha \in \mathbb{N}$, let $\|\alpha\|$ denote $\frac{\alpha}{|\alpha|}$ and note that $\|\alpha\| \in \Delta_n$. The original proof of Pólya's Theorem is "coefficient by coefficient": For $p > 0$ on Δ_n, a sequence of real polynomials p_ϵ is constructed which converge uniformly to p on Δ_n, such that the coefficient of X^α in $(\sum X_i)^N p$ is a positive multiple of $p_\epsilon(\|\alpha\|)$, where $\epsilon = \frac{1}{N+d}$. Using this technique, we can obtain "local" versions of the theorem, by which we mean the result for coefficients which correspond to exponents α such that $\|\alpha\|$ lies in a given closed subset of Δ_n. To prove our main theorem, we will write Δ_n as a union of closed subsets so that we can apply one of the local versions to each of the subsets.

The key to our local versions of Pólya's Theorem and the bounds we obtain is the simple observation that the main theorem in [7] generalizes immediately to subsets of Δ_n on which the form is positive.

If $|\alpha| = d$, define $c(\alpha) := \frac{d!}{\alpha_1! \cdots \alpha_n!}$. Suppose $p \in \mathbb{R}[X]$ is homogeneous of degree d, then write

$$p(X) = \sum_{|\alpha|=d} a_\alpha X^\alpha = \sum_{|\alpha|=d} c(\alpha) b_\alpha X^\alpha,$$

and let $L(p) := \max\limits_{|\alpha|=d} |b_\alpha|$. The following local result, which is in [6], is immediate from the proof of Theorem 1 in [7]:

Proposition 2. *Suppose $S \subseteq \Delta_n$ is nonempty and closed and $p \in \mathbb{R}[X]$ is homogeneous of degree d such that $p(x) > 0$ for all $x \in S$. Let λ be the minimum of p on S. Then for*

$$N > \frac{d(d-1)}{2} \frac{L(p)}{\lambda} - d$$

and $\alpha \in \mathbb{N}^n$ such that $\|\alpha\| \in S$, the coefficient of X^α in $(X_1 + \ldots + X_n)^N p$ is nonnegative.

The above theorem will give us an N with a bound for the region of the simplex away from the zeros. Then we will apply local results which work for closed subsets of Δ_n whose union contains the zero set of the form. We start with some notation for certain closed subsets of Δ_n containing subsets of a given face.

Definition 1. Let $F = F(I)$ be a face of Δ_n.

1. For $0 < \epsilon < 1$, let $\Delta(F, \epsilon)$ denote the following closed subset of Δ_n containing F:

$$\left\{(x_1, \dots, x_n) \in \Delta_n \mid \sum_{i \in I} x_i \leq \epsilon\right\} = \left\{(x_1, \dots, x_n) \in \Delta_n \mid \sum_{i \in I^c} x_i \geq 1 - \epsilon\right\}.$$

2. We need notation for closed subsets containing the "middle" of F, i.e., the part of F away from the lower dimensional faces. Given $0 < \tau < 1$ and $0 < \epsilon < \tau$, let

$$C(F, \epsilon, \tau) := \{(x_1, \dots, x_n) \in \Delta(F, \epsilon) \mid x_i \geq \tau - \epsilon \text{ for } i \in I^c\}.$$

3. Given $0 < \tau < 1$, define the following closed subset of the relative interior of F:

$$W(F, \tau) := \{(x_1, \dots, x_n) \in F \mid x_i \geq \tau \text{ for } i \in I^c\}$$

.

Remark 2. It is easy to check that if F is a face of dimension $k \geq 2$, and $F_1, \dots, F_k$ are the subfaces of F of dimension $k - 1$, then

$$\Delta(F, \epsilon) \subseteq C(F, \epsilon, \tau) \cup \Delta(F_1, \tau) \cup \dots \cup \Delta(F_k, \tau)$$

The following proposition is a local result for closed neighborhoods of vertices of the simplex, which follows immediately from the proof of Proposition 2 in [6]. For $1 \leq i \leq n$, let v_i denote the i-th vertex of Δ_n, i.e., $v_i = F(I)$, where $I = \{1, \dots, i-1, i+1, \dots, n\}$.

Proposition 3. *Suppose p is a form of degree d such that $p \geq 0$ on Δ_n. Let $F = v_i$, and suppose that for every $\beta \in \Lambda^-(p)$ there is some $\alpha \in \Lambda^+(p)$ such that $\alpha_F \preceq \beta_F$. Then there is $\epsilon > 0$ and $N \in \mathbb{N}$ such that for every $\gamma \in \mathbb{N}$ with $|\gamma| = N + \deg p$ and $\|\gamma\| \in \Delta(F, \epsilon)$, the coefficient of X^γ in $(\sum X_i)^N p$ is nonnegative.*

In particular, if $p = \sum a_\alpha X^\alpha$, let c be the minimum of $\{a_\alpha \mid \alpha \in \Lambda^+(p)\}$, $d = \deg(p)$, and $U = \sum |a_\alpha|$, then this holds for

$$\epsilon = \frac{c}{c+U}, \qquad N > \frac{d(d-1)}{2} \frac{L(p)}{s}, \qquad \text{where} \qquad s = \frac{c}{2}\left(\frac{2U}{c+2U}\right)^2.$$

Finally, we need a localized Pólya's Theorem which holds on the closed subsets $C(F, \epsilon, \tau)$ defined above. This result, without the explicit bound, is a special case of [6, Proposition 1].

Lemma 1. *Suppose $F = F(I)$ is a face of Δ_n, and $\phi, \psi \in \mathbb{R}[X]$ are forms of the same degree d such that*

1. $\phi > 0$ on the relative interior of F,

2. all monomials in ϕ have no factors X_i for $i \in I$, and

3. every monomial in ψ contains at least one factor X_i for some $i \in I$.

Given any $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}$ such that $\beta_j = 0$ for all $j \in I^c$ and let $p = X^\beta(\phi + \psi)$. Then for any $\tau \in \mathbb{R}$ with $0 < \tau < 1/n$, there is $0 < \epsilon < \tau$ and $N \in \mathbb{N}$ such that the coefficient of X^α in $(\sum X_i)^N p$ is nonnegative for any $\|\alpha\| \in C(F, \epsilon, \tau)$.

In particular, let λ be the minimum of ϕ on the closed subset $W(F, \tau/2)$ of the relative interior of F, and let U be the sum of the absolute value of the coefficients of p. Then this holds for

$$N > d(d-1)\frac{L(p)}{\lambda} - d,$$

and

$$\epsilon < \min\left\{\frac{\lambda}{2d\lambda + 2U}, \tau\right\}.$$

Proof. Let $q = \phi + \psi$. Claim: For ϵ and λ as given, $q \geq \lambda/2$ on $C(F, \epsilon, \tau)$
Proof of claim: Fix ϵ with $0 < \epsilon < \tau/2$ and let $C = C(I, \epsilon, \tau)$. We want to bound ϕ and ψ on C. Given $x = (x_1, \dots, x_n) \in \Delta(F, \epsilon)$ and suppose $\gamma \in \operatorname{supp}(\psi)$. Then $x_i \leq \epsilon$ for $i \in I$ and X^γ contains a factor X_i for some $i \in I$, hence X^γ evaluated at x is $\leq \epsilon$. It follows that $|\psi(x)| \leq U\epsilon$ on $\Delta(F, \epsilon)$. Since $C \subseteq \Delta(F, \epsilon)$, we have $\psi \geq -U\epsilon$ on C.

Given $x \in C$, we have $\sum\limits_{j \in I^c} x_j = 1 - t$ for $t = \sum\limits_{i \in I} x_i \leq \epsilon$. Define $a = (a_1, \dots, a_n)$ by setting $a_i = \dfrac{x_j}{1-t}$ for $j \in I^c$ and $a_i = 0$ for $i \in I$. Then $a \in F$ and for $j \in I^c$ we have $a_j = \dfrac{x_j}{1-t} > x_j \geq \tau - \epsilon \geq \tau/2$, hence $a \in W$. Then, since ϕ is a form of degree d, $\phi(x) = (1-t)^d \phi(a) \geq (1-\epsilon)^d \lambda$.

Putting together the two bounds, we have for $x \in C$,

$$q(x) \geq (1-\epsilon)^d \lambda - U\epsilon \geq (1 - d\epsilon)\lambda - U\epsilon = \lambda - (d\lambda + U)\epsilon.$$

Hence with ϵ as given we have $q(x) \geq \lambda - \lambda/2 = \lambda/2$, and the claim is proven.

Now, by Proposition 2, for the given N we have that the coefficient of X^α in $(\sum X_i)^N q$ is nonnegative for $\|\alpha\| \in C$. We just need to show that this still holds if we replace q by $p = X^\beta q$. Suppose $\gamma \in \operatorname{supp}((\sum X_i)^N p)$ with $\|\gamma\| \in C$, then $\gamma = \alpha + \beta$ with $\alpha \in \operatorname{supp}((\sum X_i)^N q)$. Since $\beta_j = 0$ for $j \in I^c$, we have $\gamma_j = \alpha_j$ for $j \in I^c$. Then since $|\gamma| \geq |\alpha|$, it is easy to see that $\gamma \in C$ implies $\alpha \in C$. It follows that the coefficient of X^α in $(\sum X_i)^N q$ is nonnegative and therefore the coefficient of X^γ in $(\sum X_i)^N p$ is nonnegative, completing the proof. □

Remark 3. The dependence of ϵ on τ is due to the fact that λ depends on τ.

4 Pólya's Theorem with Zeros

In this section we give necessary and sufficient conditions for a form to be in $Po(n,d)$ and a bound on the exponent N needed. One condition is the necessary condition (3) from Proposition 1 for the faces of Δ_n in $Z(p)$. The second condition involves the positivity of certain forms related to p on the relative interior of the faces in $Z(p)$.

For a form p and a face F of Δ_n, the relation $\alpha_F \preceq \beta_F$ defines a partial order on $\Gamma^+(p)$. We say $\alpha \in \Lambda^+(p)$ is *minimal with respect to* F if α is a minimal element in this partial order, i.e., if there is no $\gamma \in \Lambda^+(p)$ such that $\gamma_F \prec \alpha_F$. We start with notation for certain subforms of p related to the elements minimal with respect to a face.

Definition 2. Suppose $p = \sum a_\alpha X^\alpha \in \mathbb{R}[X]$.

1. For $\Gamma \subseteq \operatorname{supp}(p)$, $p(\Gamma)$ denotes the form $\sum_{\gamma\in\Gamma} a_\gamma X^\gamma$.

2. For $\alpha \in \operatorname{supp}(p)$ and a face F of Δ_n, define

$$q(\alpha, F) := p(\{\gamma \in \operatorname{supp}(p) \mid \gamma_F = \alpha_F\})/X^{\alpha_F}.$$

   Note that $q(\alpha, F)$ is a form in the variables $\{X_j \mid j \in I^c\}$, where $F = F(I)$, and $X^{\alpha_F} q(\alpha, F)$ is a subform of p.

Lemma 2. *Suppose $p \in Po(n,d)$ and F is a face of Δ_n. Then for every $\alpha \in \Lambda^+(p)$ which is minimal with respect to F, the form $q(\alpha, F)$ must be strictly positive on the relative interior of the face F.*

Proof. Suppose $F = F(I)$ and set $q := q(\alpha, F)$. Since $p \in Po(n,d)$, there exists $N \in \mathbb{N}$ such that $(\sum_{i=1}^n X_i)^N p \in \mathbb{R}^+[X]$. We claim that $(\sum_{j\in\bar{I}} X_j)^N X^{\alpha_F} q \in \mathbb{R}^+[X]$.

Suppose γ is in the support of $(\sum_{j\in\bar{I}} X_j)^N X^{\alpha_F} q$, then by the definition of q, $\gamma_F = \alpha_F$. Consider the coefficient of X^γ in $(\sum_{i=1}^n X_i)^N p$ and let A_γ be the positive part and B_γ the negative part, as in (1) and (2) in §2. Contributions to A_γ come from $\delta \in \Lambda^+(p)$ with $\delta \preceq \gamma$, which implies $\delta_F \preceq \gamma_F = \alpha_F$. Since α is minimal with respect to F, it follows that the only contributions to A_γ come from $\delta \in \Lambda^+(p)$ with $\delta_F = \alpha_F$. Since all such δ are in $\operatorname{supp}(X^{\alpha_F} q)$, it follows that A_γ is the positive part of the coefficient of X^γ in $(\sum_{j\in\bar{I}} X_j)^N X^{\alpha_F} q$. Since $X^{\alpha_F} q$ is a subform of p, the negative part of the coefficient of $(\sum_{j\in\bar{I}} X_j)^N X^{\alpha_F} q$ is clearly $\leq B_\gamma$ and it follows that the coefficient of X^γ in $(\sum_{j\in\bar{I}} X_j)^N X^{\alpha_F} q \geq A_\gamma - B_\gamma \in \mathbb{R}^+$.

From the claim it follows that $(\sum_{j\in\bar{I}} X_j)^N q \in \mathbb{R}^+[X]$. Since q is a form in $\{X_j \mid j \in \bar{I}\}$, this means that q satisfies Pólya's Theorem on the simplex $F(I)$. Hence, by Proposition 1, q is strictly positive on the relative interior of F. $\square$

Theorem 2. *Given p, a form of degree d, such that $p \geq 0$ on Δ_n and $Z(p) \cap \Delta_n$ is a union of faces. Then $p \in Po(n,d)$ if and only if for every face $F \subseteq Z(p)$ the following two conditions hold:*

1. *For every $\beta \in \Lambda^-(p)$, there is $\alpha \in \Lambda^+(p)$ so that $\alpha_F \preceq \beta_F$.*

2. *For every $\alpha \in \Lambda^+(p)$ which is minimal with respect to F, the form $q(\alpha, F)$ is strictly positive on the interior of F.*

Proof. Condition (1) is necessary by Proposition 1 and condition (2) is necessary by Lemma 2.

Suppose the conditions hold for $p = \sum a_\alpha X^\alpha \in \mathbb{R}[X]$. Assuming p is not identically zero, by Lemma 1, any face in $Z(p) \cap \Delta_n$ has dimension $\leq n-2$. For a closed set of Δ_n which does not contain the zero set, we can apply Proposition 2, hence the theorem will follow easily from the following claim:

Claim: For every face F contained in $Z(p) \cap \Delta_n$ we can find $0 < \epsilon < 1$ and $N \in \mathbb{N}$ so that for any $\theta \in \mathbb{N}^n$ with $|\theta| = N + d$ and $\|\theta\| \in \Delta(I, \epsilon)$, the coefficient of X^θ in $(\sum_{i=1}^n X_i)^N p$ is nonnegative.

Proof of claim: By induction on the dimension of F. If the dimension is 1, then $\alpha_F \preceq \beta_F$ with $\alpha \neq \beta$ implies $\alpha_F \prec \beta_F$, and we are done by Proposition 3.

Now suppose F has dimension k, $2 \leq k \leq n-2$, and the claim is true for all subfaces of F of dimension $k-1$. Let τ be the minimum of $1/n$ and the ϵ's that occurs among these subfaces and $\tilde{N}$ the maximum of the N's.

By assumption, for each $\beta \in \Lambda^-(p)$, there is $\alpha \in \Lambda^+(p)$ such that $\alpha_F \preceq \beta_F$. Clearly there is such an α which is minimal with respect to F. Order the set of $\alpha \in \Lambda^+(p)$ which are minimal with respect to F in some way and, one at a time, define forms ψ_α as follows: Let Γ_α be the set of $\beta \in \Lambda^-(p)$ such that $\alpha_F \prec \beta_F$ and β is not contained in any previously defined Γ_α. Now let $\psi_\alpha = p(\Gamma_\alpha)/X^{\alpha_F}$. Then ψ_α is a form and every monomial of ψ_α contains at least one variable X_i for $i \in I$.

Now, for each α minimal with respect to F, let $\phi_\alpha = q(\alpha, F)$ and consider the subform $X^{\alpha_F}(\phi_\alpha + \psi_\alpha)$ of p. By assumption, $q(\alpha, F)$ is strictly positive on the interior of F. Hence ϕ_α, ψ_α, and α_F satisfy the conditions of Proposition 1 and thus there is some $N_\alpha \in \mathbb{N}$ and $\epsilon_\alpha > 0$ such that the coefficient of X^γ in $(\sum X_i)^{N_\alpha} X^{\alpha_F}(\phi_\alpha + \psi_\alpha)$ is nonnegative for all $\gamma \in \mathbb{N}^n$ with $\|\gamma\| \in C(F, \epsilon_\alpha, \tau)$.

By construction, for every $\beta \in \Lambda^-(p)$, the term $a_\beta X^\beta$ in p occurs in $X^{\alpha_F}(\phi_\alpha + \psi_\alpha)$ for some α minimal with respect to F. Hence we can write

$$p = \sum X^{\alpha_F}(\phi_\alpha + \psi_\alpha) + \tilde{p},$$

where the sum is over that set of α's minimal with respect to F, and $\tilde{p}$ has only positive coefficients. Let ϵ be the minimum of the ϵ_α's and $\tau/2$, and let M be the maximum of the N_α's, then for $\gamma \in \operatorname{supp}((\sum X_i)^M p$ with $\gamma \in C(F, \epsilon, \tau)$, the coefficient of X^γ is nonnegative. Since $\Delta(F, \epsilon) \subseteq C(F, \epsilon, \tau) \cup \Delta(F_1, \tau) \cup \cdots \cup \Delta(F_k, \tau)$, the claim now follows.

Now write $Z(p) \cap \Delta_n$ as a union of faces $F_1 \cup \cdots \cup F_l$, where $F_i \not\subseteq F_j$ for any $i \neq j$, and apply the claim to each F_i, say we have that the claim holds with ϵ_i and N_i. Let S be the closure of $\Delta_n \setminus \cup_{i=1}^l \Delta(F, \epsilon_i)$, then $p > 0$ on S. By Lemma 2 there is M such that for every $\theta \in \mathbb{N}$ with $\|\theta\| \in S$ the coefficient of X^θ in $(\sum X_i)^M p$ is nonnegative. Taking the maximum of M and the N_i's, we are done.

□

The sufficient condition for $p \in Po(n,d)$ given in [4] follows easily from the theorem.

Corollary 1. *Suppose p is a form of degree d with $p \geq 0$ on Δ_n and $Z(p) \cap \Delta_n$ is a union of faces. Suppose further that for every face $F \subseteq Z(p)$ and every $\beta \in \Lambda^-(p)$, there exists $\alpha \in \Lambda^+$ such that $\alpha_F \prec \beta_F$.*

Proof. If the given condition holds for p, then the first condition of Theorem 2 holds trivially. For every α which is minimal with respect to F, by the given condition, there is no $\beta \in \Lambda^-(p)$ such that $\beta_F = \alpha_F$. Hence every $q(\alpha, F)$ has only positive coefficients and thus must be strictly positive on the interior of F. By Theorem 2, this implies $p \in Po(n,d)$. □

We now give a bound on the exponent N needed in Theorem 2. The bound will depend on the degree of p, the size of the coefficients, and constants which are defined recursively in terms of minimums of the $q(\alpha, F)$'s on a certain closed subset of the interior of F. We begin with the definition of these constants.

Definition 3. Suppose $p \in \sum a_\alpha X^\alpha$ is a form of degree d and F is a face of Δ_n such that either F is a vertex or p satisfies (2) of Theorem 2 on every subface G (including $G = F$), i.e., for every $\alpha \in \Lambda^+(p)$ which is minimal with respect to G, the form $q(\alpha, G)$ is strictly positive on the interior of G. Suppose $\dim F = k$, then we define constants $\epsilon_i(F)$, $\lambda_i(F)$ for $i = 1, \ldots, k$, recursively as follows:

$$\epsilon_1 = \min\left\{\frac{c}{c+U}, \frac{1}{n}\right\}, \quad \lambda_1 = \frac{c}{2}\left(\frac{2U}{c+2U}\right)^2,$$

where $c = \min\{a_\alpha \mid \alpha \in \Lambda^+(p)\}$ and $U = \sum_{\alpha \in \mathrm{supp}(p)} |a_\alpha|$.

For $2 \leq i \leq k$, for each subface G of F of dimension i, let $\lambda(G)$ be the minimum over all α minimal with respect to G of the minimum of $q(\alpha, G)$ on $W(G, \epsilon_{i-1}/2)$. Now let λ_i be the minimum over all G's of the $\lambda(G)$'s and choose

$$\epsilon_i < \min\left\{\frac{\lambda_i}{2d\lambda_i + 2U}, \epsilon_{i-1}\right\}. \tag{4}$$

Finally, let $\epsilon(F) = \epsilon_k(F)$ and $\lambda(F) = \lambda_k(F)$.

Theorem 3. *Given $p \in Po(n,d)$, then $Z(p) \cap \Delta_n$ is a union of faces of Δ_n. Write $Z(p) \cap \Delta_n = F_1 \cup \cdots \cup F_l$, where each F_i is a face and $F_i \not\subseteq F_j$ for all $i \neq j$. Let ϵ be the minimum of $\{\epsilon(F_i)\}$ and λ the minimum of $\{\lambda(F_i)\}$. Then $(\sum X_i)^N p \in \mathbb{R}^+[X]$ for*

$$N > \frac{d(d-1)}{2} L(p) \cdot \max\left(\frac{1}{\lambda}, \frac{1}{\theta}\right),$$

where θ is the minimum of p on the closure of $\Delta_n \setminus (\Delta(F_1, \epsilon) \cup \cdots \cup \Delta(F_l, \epsilon))$.

Proof. This follows from the proof of Theorem 2 and the bound in Lemma 1. □

Example. We continue with our example from Section 2. For $0 < a \leq 2$, we have

$$p = X_1^4 + X_2^4 + X_1^2X_3^2 + X_1^2X_4^2 - aX_1^2X_3X_4.$$

Recall $Z(p) \cap \Delta_n$ is the face $F = F(\{1,2\})$. There are two elements of Λ^+ which are minimal with respect to F: $(2,0,2,0)$ and $(2,0,0,2)$. In both cases, the form $q(\alpha, F) = q := X_3^2 + X_4^2 - aX_3X_4$. Note that $q > 0$ on the relative interior of F iff $a < 2$ and hence Theorem 2 says that $p \in Po(n,d)$ iff $a < 2$, as claimed in Section 2.

We now compute the bound from Theorem 3. We are interested in the behavior as $a \to 2$, hence there is no harm in assuming $a \geq 1$. The first step is to compute the constants $\epsilon = \epsilon(F)$ and $\lambda = \lambda(F)$ from Definition 3. We have $L(p) = 1$, $c = 1$, and $U = 4 + a$, hence $\epsilon_1 = \frac{1}{5+a}$ and $\lambda_1 = \frac{1}{2}(\frac{8+2a}{9+2a})^2$.

Next we need to find λ_2, which is the minimum of q on $W(F, \epsilon_1/2)$. It's easy to check q has a local minimum of $\frac{2-a}{4}$ at $(0,0,1/2,1/2)$ and this is the global minimum. Hence $\lambda(F) = \lambda_2 = \frac{2-a}{4}$. We need ϵ_2 satisfying (4), suppose we have chosen an appropriate $\epsilon_2 = t$. Now we need to find θ, the minimum of p on S, the closure of $\Delta_4 \setminus \Delta(F,t)$. There are no critical points in the interior of Δ_n, hence the minimum occurs on the boundary of S. Suppose $X_3 + X_4 = t$ and $X_1 + X_2 = 1 - t$. An easy calculation shows that the minimum of $X_3^2 - aX_3X_4 + X_4^2$ is $\frac{(2-a)t^2}{4}$ and this occurs when $X_3 = X_4 = t/2$. Thus we need to find the minimum of $X_1^4 + X_2^4 + X_1^2\left(\frac{(2-a)t^2}{4}\right)$. Clearly, the $X_1^4 + X_2^4$ term will dominate, and for small enough t, we will have $\max\left(\frac{1}{\lambda}, \frac{1}{\theta}\right) = \frac{4}{2-a}$, which yields the bound

$$N > \frac{24}{2-a},$$

which, as $a \to 2$, is asymptotically the same as the smallest exponent $4(2-a)^{-1} + \mathcal{O}(1)$ claimed in Section 2.

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