4-2008

A Closer Look at the Crease Length Problem

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NOTES

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An optimization problem that appears as an exercise in most modern calculus textbooks (Larson [**3**] and Stewart [**5**] for example) is the "crease length problem":

> One corner of a rectangular piece of paper with dimensions $a \times b$ (where a and b are given with $0 < a \leq b$) is folded to a point on the long side of the paper (the side of length b) and the fold is then flattened to form a crease. What is the minimum possible length of such a crease and to what point on the long side of the paper must the corner be folded in order to achieve this minimum?

Although not always stated as such, the problem that the textbooks authors actually intend for their readers to solve is a more restricted version of the problem stated above. Upon consulting the solutions manuals that accompany many of the textbooks, we find that the solutions to the crease length problem that are provided only take into account those paper foldings that do not produce a flap that protrudes over one of the edges of the paper. However, as we can convince ourselves by grabbing a piece of paper and doing some folding experiments, some of the possible folds (as described in the problem above) do produce protruding flaps. Specifically, referring to FIGURE 1, we can perform a Case 1 fold which produces a flap that protrudes over the short edge of the paper, a Case 2 fold which has no protrusion, or a Case 3 fold which produces a flap that protrudes over the long edge of the paper. In addition, there are two "critical" folds, illustrated in FIGURE 2, that separate Case 1 from Case 2 and Case 2 from Case 3, and there are also two other critical folds (not illustrated)—folding the lower

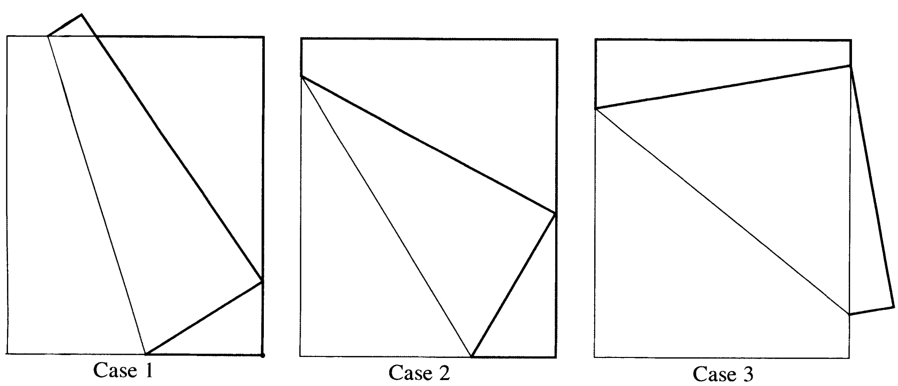

Figure 1 Three possible folds

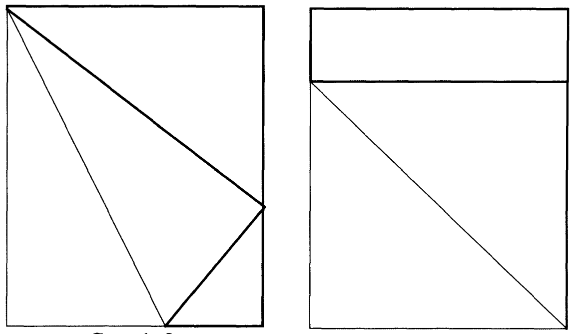

Figure 2 Critical cases

left corner onto the lower right corner (thus folding the paper in half) and folding the lower left corner onto the upper right corner (which can be viewed as an extreme case of Case 3). In [**1**], Haga considers a much wider array of possibilities for making a single fold of a rectangular sheet of paper but the focus is on studying the areas and ratios of side lengths of the polygons that are formed by such folds rather than on determining optimal crease lengths.

In this note, we provide a solution of the general crease length problem in which all possible foldings of a corner to the opposite edge (as described above) are taken into account. One of our findings will be that the minimum crease length is never produced by a Case 2 fold (no matter the dimensions of the paper) and hence that the general crease length problem always yields a different minimum than the constrained problem that is treated in the textbooks. Our more interesting discovery, however, will be a criterion that determines which foldings must be performed in order to achieve the minimum (and maximum) crease lengths. This criterion, which does not manifest itself when only the constrained problem is considered, is a condition relating the paper dimensions to the *Golden Ratio*, which is the number $\phi = \left(1 + \sqrt{5}\right)/2$. This number is one of the "special" constants of mathematics (like π and e) that seems to show up frequently, often when least expected, in investigations of many different phenomena (geometric and otherwise). For those who would like to become better acquainted with the Golden Ratio, we recommend the book of Huntley [**2**] and the article of Markowsky [**4**].

Supposing our paper to have dimensions $a \times b$ where $0 < a \leq b$, we can view the crease length as a function of y, where y is the distance from the lower right corner of the paper to the point on the right edge of the paper to which the lower left corner has been folded. (Refer to FIGURE 1.) Our goal is to determine the absolute minimum and maximum values of the crease length and the values of y at which these extrema occur as y ranges from 0 to b. In order to construct the function that gives the length of this crease, we will find it convenient to first consider a slightly different folding problem in which the paper to be folded has no top or right boundaries.

The crease function for infinite paper

If we begin with an infinite piece of paper or "infinite open rectangle" $R = (0, \infty) \times (0, \infty)$ and fold the point $(0, 0)$ (which is the lower left corner of the rectangle) onto an arbitrary point $(x, y) \in R$, then the fold cannot protrude over any of the boundaries of

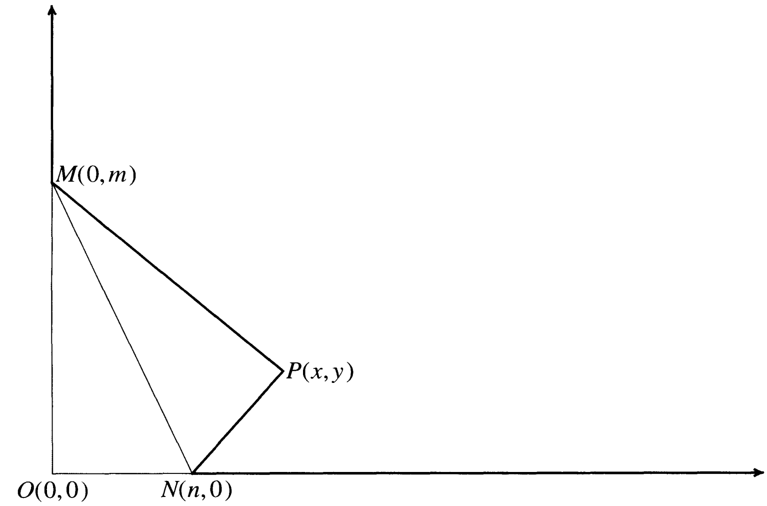

Figure 3 Folding infinite paper

the original rectangle and we obtain a situation as depicted in FIGURE 3. By referring to this figure, the crease length $|MN|$ can be determined as a function of x and y.

Since $|NP| = |ON|$, we see that $(x-n)^2 + y^2 = n^2$ and hence that

$$n = \frac{x^2 + y^2}{2x}. \tag{1}$$

Likewise, the equality of $|MP|$ and $|OM|$ implies that

$$m = \frac{x^2 + y^2}{2y}. \tag{2}$$

Then, by the Pythagorean Theorem, we find that the square of the crease length (for convenience we will always square the crease length) is given by the function

$$F(x, y) = |MN|^2 = n^2 + m^2 = \frac{(x^2+y^2)^2}{4x^2} + \frac{(x^2+y^2)^2}{4y^2} = \frac{(x^2+y^2)^3}{4x^2y^2}. \tag{3}$$

Before moving on to examine creases of finite paper, we pause to make an observation that might be of interest to students and teachers of multivariable calculus. A major topic in multivariable calculus is the study of indeterminate form limits of functions of two variables. Many of the examples and exercises through which students learn about this topic involve rational functions (ratios of polynomials in x and y). However, examples of these types of limit problems for which physical or geometric intuition can be brought to bear seem to be rare. The function F defined in (3) does provide such an example though. Specifically, let us consider the problem of evaluating $\lim_{(x,y)\to(0,0)} F(x, y)$.

The standard method of evaluating this limit is to let (x, y) approach $(0, 0)$ along various curves (such as $y = x$, $y = x^2$, $y = x^3$) and to observe that each curve of approach yields a different limit $(0, 1/4, \infty)$, thus allowing us to conclude that $\lim_{(x,y)\to(0,0)} F(x, y)$ does not exist. However, our intuitive understanding of why this limit does not exist is greatly enhanced by referring to FIGURE 3 in which $F(x, y)$ is the square of the length of the crease MN. In particular, it is easy to visualize that if we let the point $P(x, y)$ approach $O(0, 0)$ along the line $y = x$, then the crease length approaches 0. On the other hand, we observe that within any prescribed distance of O we can find points $P(x, y)$ that yield arbitrarily large crease lengths (obtained by choosing $P(x, y)$ close enough to the boundary of the paper). This visual reasoning provides us with a geometry–based understanding of why the limit in question does not exist and also suggests that polar coordinates should be useful for the purpose of obtaining a more detailed mathematical description of the behavior of F. Indeed, by letting $x = r\cos(\theta)$ and $y = r\sin(\theta)$, we see that the level curve $F(x, y) = K^2$ (corresponding to a given crease length $K > 0$) is $r = K\sin(2\theta)$, $0 < \theta < \pi/2$. The fact that each of these level curves (for each $K > 0$) lies in the open first quadrant and intersects every neighborhood of $(0, 0)$ shows that $\lim_{(x,y)\to(0,0)} F(x, y)$ does not exist and, furthermore, shows that F assumes every positive value in every neighborhood of $(0, 0)$.

The crease function for finite paper

In order to derive the crease function for a finite piece of paper of dimensions $a \times b$ where $0 < a \le b$, we position the lower left corner of our paper at the point $(0, 0)$ and the lower right corner at the point $(a, 0)$. The function that we want to derive is

$$L(y) = \text{square of the crease length when } O(0, 0) \text{ is folded onto } P(a, y)$$

with domain $0 \le y \le b$.

Since the finite rectangle $(0, a) \times (0, b)$ is a subset of the infinite rectangle $(0, \infty) \times (0, \infty)$, we will be able to make use of the infinite paper crease function, F, in deriving L. In fact, the work has already been done for a Case 2 fold (see FIGURE 1) since this type of fold produces a crease that has the same length as the crease that would be formed in folding an infinite piece of paper. Thus, for a Case 2 fold, we have by (3) that

$$L(y) = F(a, y) = \frac{(a^2 + y^2)^3}{4a^2y^2}.$$

The derivation of L for Cases 1 and 3 requires a little additional geometry. In these cases, we regard the finite paper as being superimposed on the infinite paper. Although only the finite paper is to be folded, we extend the lines formed by the fold onto the infinite paper as shown in FIGURES 4 and 5. It then follows from (1) and (2) that

$$n = \frac{a^2 + y^2}{2a} \qquad \text{and} \qquad m = \frac{a^2 + y^2}{2y}.$$

For Case 1 (FIGURE 4), the crease length is $|RN|$. The triangles MBR and MON are similar, so we obtain

$$\frac{|MR| + |RN|}{|OM|} = \frac{|MR|}{|BM|}$$

which gives us

$$|RN| = \frac{|MR|}{|BM|}|OB| = \frac{|MN|}{|OM|}|OB|$$

and thus

$$L(y) = |RN|^2 = \frac{m^2 + n^2}{m^2}b^2 = \left(1 + \frac{n^2}{m^2}\right)b^2 = \left(1 + \frac{y^2}{a^2}\right)b^2 = \frac{b^2}{a^2}(a^2 + y^2).$$

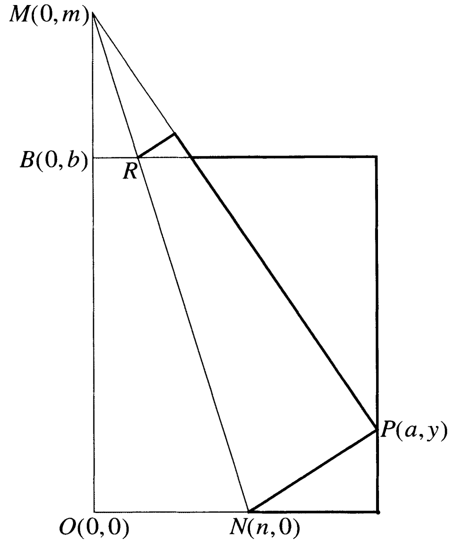

Figure 4 Case 1 fold

For Case 3 (FIGURE 5), the crease length is $|MR|$ and since

$$\frac{|MR|+|RN|}{|ON|}=\frac{|RN|}{|AN|}$$

which implies that

$$|MR|=\frac{|RN|}{|AN|}|OA|=\frac{|MN|}{|ON|}|OA|,$$

we obtain

$$L(y)=|MR|^2=\frac{m^2+n^2}{n^2}a^2=\left(\frac{m^2}{n^2}+1\right)a^2=\left(\frac{a^2}{y^2}+1\right)a^2=\frac{a^2}{y^2}(a^2+y^2).$$

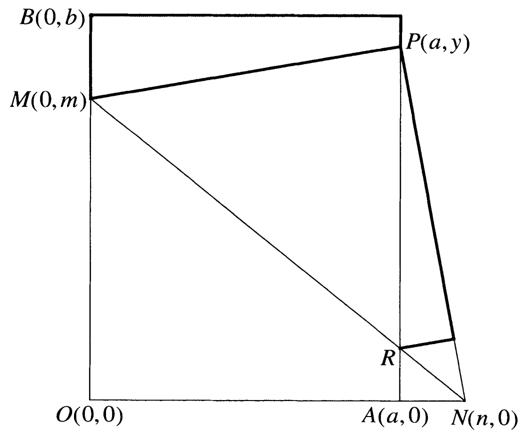

Figure 5 Case 3 fold

By comparing FIGURES 2, 4, and 5, we observe that the condition that corresponds to the critical Case 1–2 is $m = b$ which is equivalent to $y = b - \sqrt{b^2 - a^2}$ and that the condition that corresponds to the critical Case 2–3 is $n = a$ which is equivalent to $y = a$. The crease function for an $a \times b$ piece of paper is thus the piecewise–defined function

$$L(y) = \begin{cases} \dfrac{b^2}{a^2}(a^2 + y^2) & \text{if } 0 \le y \le b - \sqrt{b^2 - a^2} \\ \dfrac{(a^2 + y^2)^3}{4a^2 y^2} & \text{if } b - \sqrt{b^2 - a^2} < y < a \\ \dfrac{a^2}{y^2}(a^2 + y^2) & \text{if } a \le y \le b \end{cases}.$$

It can readily be seen that L is increasing on the interval $\left(0, b - \sqrt{b^2 - a^2}\right)$ and decreasing on the interval (a, b). On the middle interval, $\left(b - \sqrt{b^2 - a^2}, a\right)$, since

$$L'(y) = \frac{(a^2 + y^2)^2}{a^2 y^3}\left(y + \frac{\sqrt{2}}{2}a\right)\left(y - \frac{\sqrt{2}}{2}a\right),$$

we observe that $y = \sqrt{2}a/2$ is a critical point of L that corresponds to a local minimum value of L if and only if $\sqrt{2}a/2 \in \left(b - \sqrt{b^2 - a^2}, a\right)$. While the relation $\sqrt{2}a/2 < a$ is certainly always true, the relation $b - \sqrt{b^2 - a^2} < \sqrt{2}a/2$ is true (as the reader can check) if and only if $b^2/a^2 > 9/8$. In all of the textbook exercises that we have seen, the paper dimensions are given to be such that $b^2/a^2 > 9/8$ and, since only Case 2 folds are addressed in these exercises, the local minimum value $L\left(\sqrt{2}a/2\right)$ is regarded as the absolute minimum value and hence as the "right answer" for the minimum crease length. However, in what follows we will see that this is in fact never the absolute minimum in the more general problem (no matter the values of a and b).

The golden ratio makes the call

In order to economize on notation, we introduce a new parameter $q = b/a$ (with the assumption that $0 < a \le b$ implying that $q \ge 1$) and we also give names to the critical points of L: $y_0 = 0$, $y_1 = b - \sqrt{b^2 - a^2}$, $y_2 = \sqrt{2}a/2$, $y_3 = a$, and $y_4 = b$. Even though y_2 is not a critical point unless $q^2 > 9/8$, it will not be necessary to treat this case separately in what follows.

We have determined that the candidates for the absolute minimum value of L are

$$L(y_0) = b^2 = q^2 a^2$$

$$L(y_2) = \frac{27}{16}a^2$$

$$L(y_4) = \frac{a^2}{b^2}(a^2 + b^2) = \left(\frac{1}{q^2} + 1\right)a^2$$

and that the candidates for the absolute maximum value of L are

$$L(y_1) = \frac{2b^3 y_1}{a^2} = 2q^3\left(q - \sqrt{q^2 - 1}\right)a^2$$

$$L(y_3) = 2a^2.$$

To see where the absolute extrema actually occur, we need to compare the values $L(y_0)$, $L(y_2)$ and $L(y_4)$ and also compare the values $L(y_1)$ and $L(y_3)$. It is in doing this that we will see the Golden Ratio,

$$\phi = \frac{1+\sqrt{5}}{2} \approx 1.618,$$

make its appearance. The property of the Golden Ratio that will be used in our comparisons is the property that it is the only positive real number that is exactly one greater than its reciprocal; that is, $\phi^{-1} + 1 = \phi$, or equivalently $\phi^2 - \phi - 1 = 0$. The other fact that will be used is the fact that $\phi < 27/16$. Our results are given in the following:

PROPOSITION 1. *Let $q = b/a$ and let ϕ denote the Golden Ratio.*

1. *If $q^2 < \phi$, then the minimum possible crease length is achieved by folding the paper in half and the maximum is achieved by performing a Case 2–3 fold (*FIGURE *2).*
2. *If $q^2 > \phi$, then the minimum possible crease length is achieved by folding the lower left corner of the paper to the upper right corner of the paper and the maximum is achieved by performing a Case 1–2 fold.*
3. *The minimum possible crease length can be achieved with two distinct foldings if and only if $q^2 = \phi$. (The same is true of the maximum possible crease length.)*

Proof. First we compare the values $L(y_0)$, $L(y_2)$, and $L(y_4)$: If $q^2 < \phi$, then

$$\frac{1}{a^2}\left(L(y_2) - L(y_0)\right) = \frac{27}{16} - q^2 > \phi - q^2 > 0$$

and

$$\frac{1}{a^2}\left(L(y_4) - L(y_0)\right) = \frac{1}{q^2} + 1 - q^2 > \frac{1}{\phi} + 1 - \phi = 0;$$

whereas if $q^2 > \phi$, then

$$\frac{1}{a^2}\left(L(y_2) - L(y_4)\right) = \frac{27}{16} - \frac{1}{q^2} - 1 > \frac{27}{16} - \frac{1}{\phi} - 1 = \frac{27}{16} - \phi > 0$$

and

$$\frac{1}{a^2}\left(L(y_0) - L(y_4)\right) = q^2 - \frac{1}{q^2} - 1 > \phi - \frac{1}{\phi} - 1 = 0.$$

The above comparisons show that L achieves its absolute minimum value at y_0 if $q^2 < \phi$ and at y_4 if $q^2 > \phi$. If $q^2 = \phi$, then L achieves its minimum at both y_0 and y_4. In no case is the minimum achieved at y_2.

We now compare the values $L(y_1)$ and $L(y_3)$. Since the quantity

$$\frac{1}{2a^2}\left(L(y_1) - L(y_3)\right) = q^4 - q^3\sqrt{q^2-1} - 1$$

is positive if and only if

$$q^3\sqrt{q^2-1} < q^4 - 1$$

which is true if and only if

$$q^6 - 2q^4 + 1 > 0,$$

and since

$$q^6 - 2q^4 + 1 = (q^2 - 1)\left(q^2 + \frac{1}{\phi}\right)(q^2 - \phi),$$

we conclude that $L(y_1) > L(y_3)$ if and only if $q^2 > \phi$. Therefore L achieves its absolute maximum value at y_1 if $q^2 > \phi$ and at y_3 if $q^2 < \phi$. If $q^2 = \phi$, then L achieves its maximum at both y_1 and y_3. ■

In order to illustrate the somewhat surprising nature of our results, let us compare the constructions of extreme crease lengths for 8.5×11 paper and 8.7×11 paper. Since these paper dimensions are not very different (probably not visible to the naked eye), intuition would lead us to believe that the extrema would be obtained by performing similar folds. However, for $a = 8.5$ and $b = 11$ we have $q^2 \approx 1.675 > \phi$ meaning that the minimum crease length is obtained by folding the lower left corner onto the upper right corner and the maximum crease length is obtained by performing a Case 1–2 fold (FIGURE 2); whereas for $a = 8.7$ and $b = 11$ we have $q^2 \approx 1.599 < \phi$ meaning that the minimum crease length is obtained by folding the paper in half and the maximum crease length is obtained by performing a Case 2–3 fold. A comparison of the crease functions, giving actual crease lengths $\sqrt{L(y)}$, for 8.5×11 paper and 8.7×11 paper is shown in FIGURE 6.

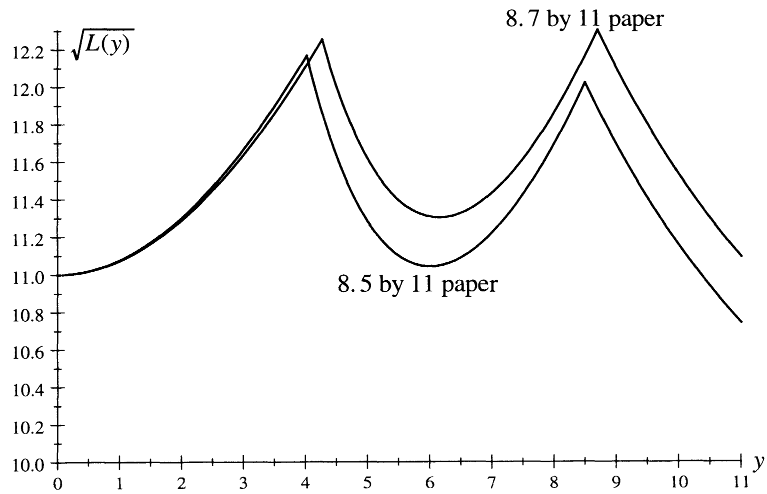

Figure 6 Comparison of crease length functions

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