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Problem solving and proving via generalisation

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"*Let us see whether we could, by chance, conceive some other general problem that contains the original problem and is easier to solve.*" - Leibnitz quoted in Polya (1954: 29)

"*As it often happens, the general problem turns out to be easier than the special problem would be if we had attacked it directly.*" - Lejeune-Dirichlet & Dedekind quoted in Polya (1954: 30)

A very useful problem solving strategy often emphasised at school and regularly tested in Mathematical Olympiads and Challenges is to consider special cases of a problem. Not only are the special cases usually more easy to solve, but often allows one to identify a pattern or give some clue towards a general solution or proof. Less well known (or utilised) appears to be the opposite strategy, namely, to consider a more general case than the given problem. Contrary to what one might expect, the general case is sometimes much easier to solve than the special case.

George Polya (1945, 1954) in his seminal problem solving books gives several examples of this strategy. For example, taking a hard problem in the plane and solving it quite easily in space, and then translating it back into the context of the plane. Similarly, a difficult numerical problem is sometimes remarkably simplified by simply translating it into algebra.

The purpose of this article is to give some illustrative examples of this strategy of solving problems and proving by generalisation. First some simple calculation examples will be given followed by some algebraic proof examples. Two geometry examples are then provided, and lastly a couple of examples of the use of complex numbers.

Some simple calculation examples

(a) Calculate $999^2 - 1$

(b) Calculate $19 \times 99 + 19$

(c) Calculate $1 + 2 + 3 + 4 + \ldots + 100$

(d) Find the 200th term in the sequence 3, 6, 11, 18, 27, 38, …

The above types of examples are quite common in AMESA Mathematics Challenges and the Harmony SA Mathematics Olympiad, and also appear in most textbooks. Since time is a factor it is clearly not sensible to just carry out the calculation as given. Moreover, the calculations are significantly simplified by transforming each of the above by observing the underlying *general structure*. Not only is time saved, but also the possibility of making calculation errors is decreased.

For example, by using the difference between squares, the first example simply becomes (999 – 1) × (999 +1) = (998) × (1000) = 998000. Similarly, the second example becomes straightforward if one transforms it using the distributive property as follows: 19 × (99 + 1) = 19 × (100) = 1900. The third example can be neatly solved by noting the following general pattern (apparently used by the young Gauss when his class was given this problem): 1 + 100 = 101; 2 + 99 = 101; 3 + 98 = 101; …; 50 + 51 = 101, and since there are 50 pairs the answer is simply 50 × 101 = 5050. Or equivalently, rewrite the series twice as follows:

1 + 2 + 3 + … + 98 + 99 + 100

<u>100 + 99 + 98 + … + 3 + 2 + 1</u>

101 + 101 + 101 + … + 101 + 101 +101 = 100 × 101

But since we've added the series to itself, the answer to the original problem is simply $100/2 \times 101$ = 5050. Note that this latter technique is but a small step away from deriving a general formula for the sum of any arithmetic series, namely, $S = \frac{n}{2}(a + l)$.

The third example shows the value and power of problem solving via general pattern or structure rather than just carrying out calculations by brute force. The next example demonstrates the utility of seeing a sequence as a function with the set of natural numbers as domain, and using algebraic techniques to obtain that function.

To find the 200th term in the sequence for the 4th example, the student could observe that the sequence of first differences is 3, 5, 7, 9, 11, … and then find the sixth term by adding 13 to the fifth term, and find the seventh term by adding 15 to the sixth term, and continue adding odd integers until you reach the 200th term. A far less tedious approach would be to find an algebraic form for the *n*th term of the sequence, and this could be accomplished in a variety of ways.

One approach could be Gauss' approach as described previously. In other words, if we view the sequence as shown in the following table, the *n*th term can be viewed as 3 plus the sum of the odd integers beginning with 3.

n	*n*th term
1	3
2	3+3
3	3+3+5
4	3+3+5+7
5	3+3+5+7+9
6	3+3+5+7+9+11
…	

So the *n*th term is 3 plus 3+5+7+9 +… + 2*n*-1. Using Gauss' approach we would obtain the following formula for the sum of the first *n*-1 terms:

$$\begin{array}{l} 3 \quad + \; 5 \quad + \; 7 \; + \quad \ldots + 2n-3 \quad + 2n-1 \\ \underline{2n-1 \; + \; 2n-3 \; + \; 2n-5 \; + \ldots \quad + 5 \quad + 3} \\ (2n+2) + (2n+2) + (2n+2) + \ldots \quad + (2n+2) + (2n+2) \end{array}$$

We could then observe that there are *n*-1 such sums, so the *n*th term of the original sequence could be obtained by the formula

$$f(x) = \frac{(n-1)(2n+2)}{2} + 3$$

Another approach relies more heavily on algebraic techniques, but can be easily generalized to sequences produced by other kinds of functions. Note that the second difference of the sequence is constant, and therefore a quadratic function could describe the sequence. Then since you know a quadratic function can be written in the form

$$f(x) = an^2 + bn + c$$

you could use a system of equations to determine the quadratic function as shown below:

$$3 = a + b + c$$
$$5 = 4a + 2b + c$$
$$7 = 3a + 9b + c$$

Solving for *a*, *b*, *c* gives the function

$$f(n) = n^2 + 2$$

The system of equations could be solved by the process of elimination of variables, but could also be solved using matrices. This approach could then be used to find a formula for sequences that have a constant third difference (cubic function), or a constant fourth difference (quartic function), etc.

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Algebraic proof

(a) Take a two-digit number and reverse its digits. Prove that the sum of these two numbers is always divisible by 11.

(b) Prove that the square of any odd number leaves a remainder of 1 when divided by 4.

(c) Prove that if the sum of the digits of a number is divisible by 3, then the number is also divisible by 3.

(d) Find a general formula for the product of the roots of a quadratic equation.

Proof by its nature has to be general and cover all cases. A couple of numerical examples are not sufficient. The power of algebra is precisely the generality obtained by replacing a specific numerical value with a variable.

Though the first example can also be done by brute force checking by hand (or computer) by starting from 10 and going up to 99, it is much easier just doing the algebra. For example, any two-digit can be written as $10a+b$ where a and b are digits from 1 to 9. Thus, $(10a+b)+(10b+a)=11(a+b)$, which is clearly divisible by 11.

Similarly, for the second example, any odd number can be written as $2n-1$. Thus, $(2n-1)^2=4n^2-4n+1=4(n^2-n)+1$, which will clearly leave a remainder of 1 if divided by 4. Note that unlike the first example this result and its proof deals with an infinite number of cases, and therefore numerical checking is quite impossible.

For the third example, consider the generic case of a three-digit number written as $100a+10b+c=(99a+9b)+(a+b+c)$. Since $99a+9b$ is clearly divisible by 3, the original three-digit number will be divisible by 3 if $a+b+c$ is also divisible by 3. In the same way this result can be proved for any number.

Let's finally consider the fourth example. Typically the familiar formula $\frac{c}{a}$ for the product of the roots, say α and β, of a quadratic equation $ax^2+bx+c=0$ is derived traditionally in many high school textbooks by using the quadratic formula for the two roots, and then multiplying them out as follows to achieve the desired result:

$$\frac{-b+\sqrt{\Delta}}{2a}\times\frac{-b-\sqrt{\Delta}}{2a}=\frac{b^2-(b^2-4ac)}{4a^2}=\frac{c}{a}.$$

The problem with this approach is that it is limited because if one wanted to find the product of the roots of a cubic or a quartic equation, one would have to firstly know the respective formulae for their roots. More-over, this approach will break down when one considers polynomials of order 5 or higher as it has been proved that no such general formula exists for their roots.

However, let's consider a general polynomial equation $a_1x^n + a_2x^{n-1} + ... + a_nx + a_{n+1} = 0$ with roots $\alpha_1, \alpha_2, ... \alpha_{n-1}, \alpha_n$. Therefore, $(x-\alpha_1)(x-\alpha_2)...(x-\alpha_{n-1})(x-\alpha_n) = 0$, which upon multiplying out clearly gives us $x^n + ... + \alpha_1\alpha_2...\alpha_{n-1}\alpha_n = 0$ if n is even or $x^n + ... - \alpha_1\alpha_2...\alpha_{n-1}\alpha_n = 0$ if n is odd. By dividing the polynomial equation through by a_1 we obtain $x^n + \frac{a_2}{a_1}x^{n-1} + ... + \frac{a_n}{a_1}x + \frac{a_{n+1}}{a_1} = 0$. By now comparing term by term the latter equation with the preceding two, it immediately follows that the product of the roots of a polynomial equation in general is $\frac{a_{n+1}}{a_1}$ if n is even and $-\frac{a_{n+1}}{a_1}$ if n is odd.

Not only do we now have a general formula for *any* polynomial, but the derivation above requires almost no algebraic manipulation nor a formula for the roots.

Solving 2D by going 3D

Normally a generalisation in mathematics, precisely because it is more general, is harder to prove than a special case. One need only think of the cosine rule as a generalisation of the theorem of Pythagoras, the arithmetic-mean/geometric-mean inequality generalised to n numbers, etc. Similarly, problems in 3 dimensions are usually much harder than their corresponding counterparts in 2 dimensions. However, this is not always true. Surprisingly, a generalisation can sometimes simplify a problem dramatically and unexpectedly, as will be strikingly illustrated by the following two examples.

The first example is Desargues's theorem, discovered by Gérard Desargues (1591-1661), a French architect and geometer, and provides a striking example of the power of sometimes generalising to three dimensions. Succinctly put, this remarkable theorem states that "*Two triangles are point perspective, if and only if, they are also line perspective*."

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As shown in Figure 1, this means that if for triangles ABC and $A'B'C'$, lines AA', BB' and CC' connecting corresponding vertices are concurrent in a point *O*, then the respective intersections *X*, *Y* and *Z* of the corresponding sides (extended if necessary) are collinear (lie on a line), and of course, conversely, the other way round. Now this theorem is quite difficult to prove in the plane only using the axioms of plane Euclidean geometry (and perhaps an intrepid reader may want to attempt it to see that it is no easy task!). However, if we consider the two triangles as lying not on the same plane, but instead on two non-parallel planes in three dimensions, the problem becomes almost trivial.

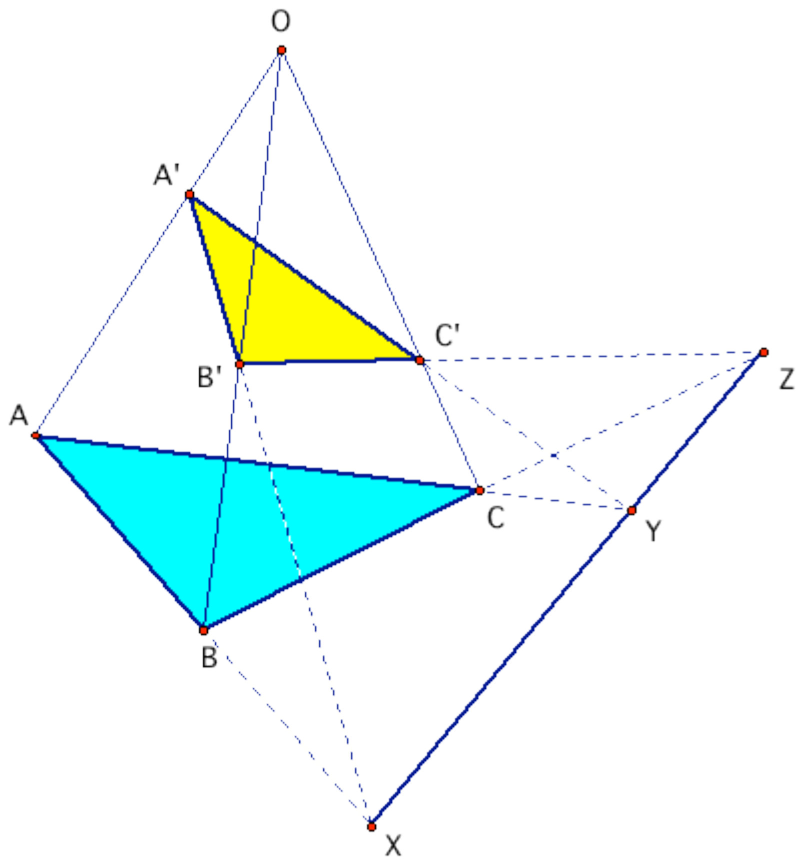

Figure 1

Proof

Let's assume the two triangles are point perspective in *O*. Each pair of corresponding sides of the two triangles are coplanar (i.e. contained in a plane). Therefore, each pair of corresponding sides, since they are coplanar and not parallel (they lie on non-parallel planes) must intersect, if extended, in the respective intersection points *X*, *Y* and *Z*. But the points *X*, *Y* and *Z* must belong to both planes ABC and $A'B'C'$, and thus to the intersection of the two planes. However, since the intersection of two non-parallel planes is a straight line, it follows that *X*, *Y* and *Z* are collinear.

The converse can be proved in the same way. Moreover, since collinearity (line-perspectiveness) and concurrency (point-perspectiveness) are always preserved in projective geometry, the two-dimensional version can now simply be considered as a projection of the three-dimensional version onto the plane!

The second example dates from the 19^{th} century and is another striking example of the generalisation heuristic. If triangles *DBA*, *ECB* and *FAC* are constructed outwardly on the sides of any Δ*ABC* so that *DA* = *FA*, *DB* = *EB* and *EC* = *FC*, then the perpendicular from *D* to *AB*, the perpendicular from *E* to *BC* and the perpendicular from *F* to *AC* are concurrent (see Figure 2).

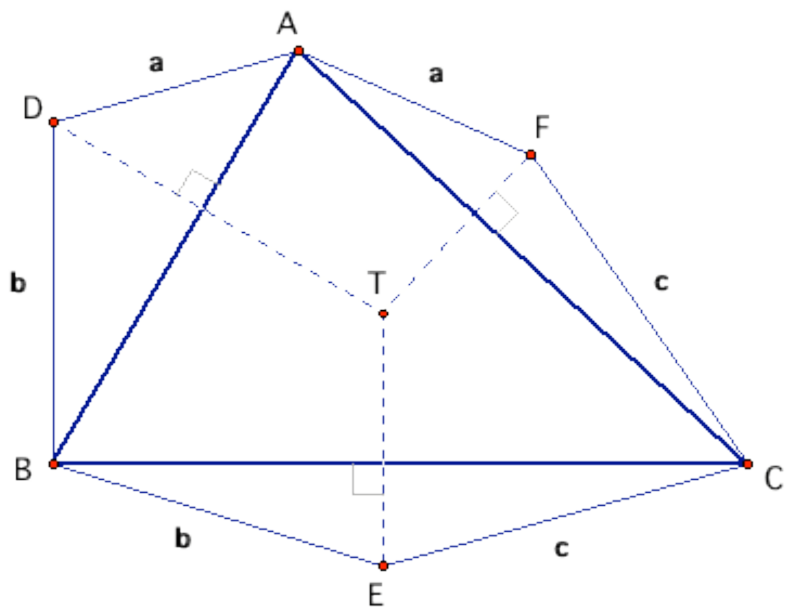

Figure 2

Like Desargues's theorem this result is very hard to prove using plane Euclidean geometry, and is usually proved in advanced geometry texts by means of inversive geometry utilising the inversive concept of the "*power of a point*". The perpendiculars *DT*, *ET* and *FT* are generally called "*power lines*" and the point *T* the corresponding "*power point*" of a triangle.

However, this result can be proved very easily by considering a tetrahedron and folding it flat as shown in Figure 2. Reconstructing the tetrahedron by folding up points *D*, *E* and *F* to meet at the top vertex *T*, it follows that the perpendiculars from *D*, *E* and *F* to the three sides must meet at the foot of the perpendicular from *T* to the plane *ABC*.

(Just a word of caution that the above argument actually does not provide a

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sufficiently general proof to cover all possibilities in the plane as it's based on the assumption that folding up any 2D-configuration like that shown in Figure 2 will give a tetrahedron. But that will only be possible when for example the sum of the two exterior angles surrounding each vertex is greater than the interior angle of the triangle at that vertex).

The preceding two results not only provide some interesting enrichment material, but could also be used to illustrate to students the value of sometimes considering a more general case of a problem at hand.

Generalizing the Real Number System

(a) Prove De Moivre's Theorem

$$\left[r(\cos(\theta)+i\sin(\theta))\right]^n = r^n(\cos(n\theta)+i\sin(n\theta))$$

(b) Given any quadrilateral, construct squares on each of the sides of the quadrilateral. Prove the line segments joining the centers of opposite squares are perpendicular and of equal length. (This result is known as Van Aubel's theorem).

The history of Mathematics could be described as the effort to identify, describe, and generalize patterns. One example is the historical march from arithmetic to school algebra to abstract algebra. Another example is the generalization of the real number system to complex numbers. It took hundreds of years for mathematicians to accept the complex numbers; indeed, the name "imaginary" for the square root of -1 was intentional and at first considered quite appropriate (Nahin, 1998). It was the geometric interpretation of complex numbers as points in the plane that opened the door to a wealth of "real" applications. In the classroom, complex numbers not only offer an interesting story historically, but can show the student how changing the setting or form of a problem can lead to simple solutions.

Proof of De Moivre's Theorem is trivial using Euler's classic formula:

$$re^{i\theta} = r(\cos(\theta)+i\sin(\theta))$$

since $$(re^{i\theta})^n = r^n(e^{i(n\theta)})$$

by the properties of exponents. Indeed, many trigonometric identities can be proven easily using Euler's formula.

Proof of the second result is challenging without complex numbers (for a transformation geometry proof of this result, see Yaglom (1962), and for transformation geometry proofs of generalizations of this result to similar rectangles and similar rhombi on the sides, see De Villiers (1998)).

Consider the quadrilateral *ABCD* in Figure 3, with sides represented by the complex numbers (or vectors) a, b, c, and d, with the initial point of a at the origin of the complex plane. The centers of the squares constructed on each of the sides are *E*, *F*, *G*, and *H*. Since the quadrilateral is a closed figure, $a + b + c + d = 0$.

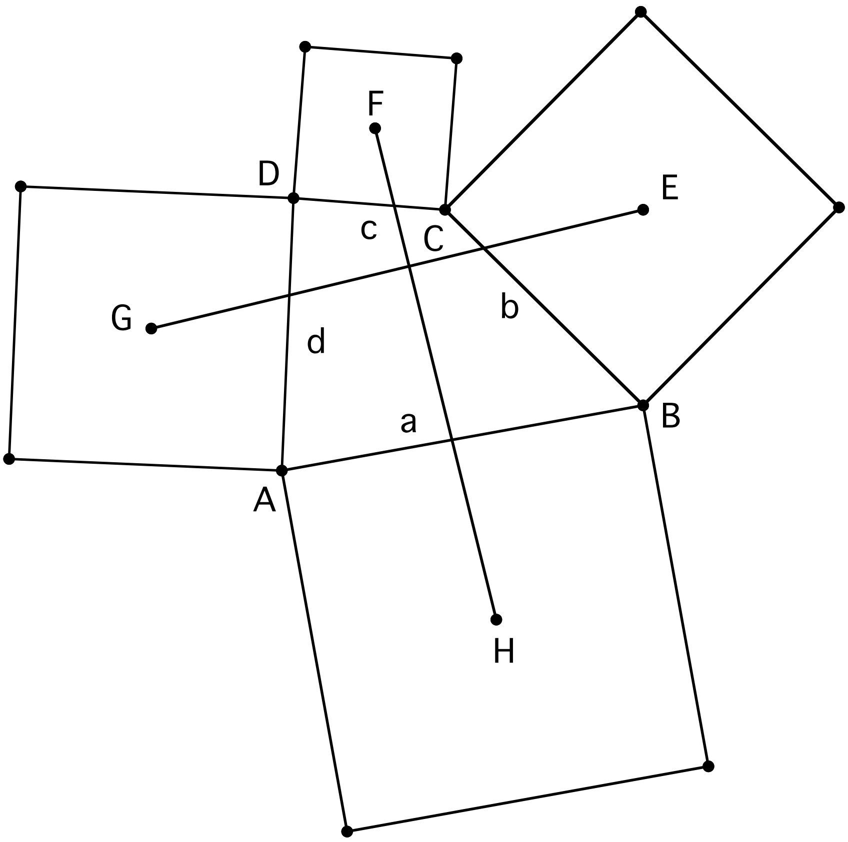

Figure 3

Since vertex *A* is at the origin of the complex plane, the location of *H* can be described as $\frac{a}{2} - i\frac{a}{2}$. The location of *E* is $a + \frac{b}{2} - i\frac{b}{2}$, and of F is $a + b + \frac{c}{2} - i\frac{c}{2}$, and of *G* is $a + b + c + \frac{d}{2} - i\frac{d}{2}$. We want to show that the line segment *FH* is perpendicular and congruent to the line segment *GE*. The segment *FH* can be described as the complex number

$$a+b+\frac{c}{2}-i\frac{c}{2}-\frac{a}{2}+i\frac{a}{2}$$
$$=b+\frac{a+c}{2}-i(\frac{c-a}{2})$$
$$=\frac{a+2b+c}{2}-i(\frac{c-a}{2})$$

and the segment through *GE* can be described as the complex number

$$a+b+c+\frac{d}{2}-i\frac{d}{2}-a-\frac{b}{2}+i\frac{b}{2}$$
$$=c+\frac{b+d}{2}-i(\frac{d-b}{2})$$
$$=\frac{b+2c+d}{2}-i(\frac{d-b}{2})$$

Since $a + b + c + d = 0$, then $a = -(b + c + d)$ and $d = -(a + b + c)$. Substituting $-a$ for $(b + c + d)$ and substituting $-(a + b + c)$ for d in the complex number representing *GE* we get:

$$\frac{b+2c+d}{2}-i(\frac{d-b}{2})$$
$$=\frac{c-a}{2}+i\left(\frac{a+2b+c}{2}\right)$$

Thus, *FH* can be described as

$$\frac{a+2b+c}{2}-i(\frac{c-a}{2})$$

and *GE* can be described as

$$\frac{c-a}{2}+i\left(\frac{a+2b+c}{2}\right)$$

The moduli of these two complex numbers are clearly the same, and *GE* can be obtained from *FH* by multiplication by $-i$, which indicates they are perpendicular.

Note

Dynamic Geometry (*Sketchpad 4*) sketches in zipped format (Winzip) of the geometry results discussed here can be downloaded directly from:
http://mysite.mweb.co.za/residents/profmd/desarguespowerlines.zip

(If not in possession of a copy of *Sketchpad 4*, these sketches can be viewed with a free

demo version of *Sketchpad 4* that can be downloaded from:
http://www.keypress.com/x17670.xml)

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