12-1988

A Simple Test for the nth Term of a Series to Approach Zero

Jonathan W. Lewin
Kennesaw State University, jlewin@kennesaw.edu

Myrtle Lewin
Agnes Scott College

Recommended Citation
Lewin, J., & Lewin, M. (1988). A simple test for the nth term of a series to approach zero. The American Mathematical Monthly, 95(10), p. 942.

THE TEACHING OF MATHEMATICS

EDITED BY MELVIN HENRIKSEN AND STAN WAGON

A Simple Test for the *n*th term of a Series to Approach Zero

JONATHAN LEWIN
Department of Mathematics, Kennesaw College, Marietta, GA 30061

MYRTLE LEWIN
Department of Mathematics, Agnes Scott College, Decatur, GA 30030

Using Stirling's formula, one may see at once that if $a_n = (2n)!/4^n(n!)^2$, then a_n is of the order of $1/\sqrt{n}$, and one may conclude from the alternating series test that the series $\Sigma(-1)^n a_n$ is conditionally convergent. At an elementary level, however, the convergence of the latter series may be a little more difficult to obtain. Since $a_{n+1}/a_n = (2n+1)/(2n+2) < 1$ for each n, it is clear that the sequence (a_n) is decreasing, but it is not immediately obvious within the environment of a typical calculus course that $a_n \to 0$ as $n \to \infty$. For this purpose, one might use the following simple result which takes a leaf out of the theory of infinite products:

THEOREM. *Suppose (a_n) is a decreasing sequence of positive numbers and for each natural number n, define $b_n = 1 - a_{n+1}/a_n$. Then the sequence (a_n) converges to zero if and only if the series Σb_n diverges.*

Proof. We note first that unless $b_n \to 0$ as $n \to \infty$, both of the series Σb_n and $\Sigma \log(1 - b_n)$ diverge. On the other hand, if $b_n \to 0$, then $b_n/(-\log(1 - b_n)) \to 1$ as $n \to \infty$, and it follows from the limit comparison test that Σb_n diverges if and only if $\Sigma \log(1 - b_n)$ diverges. We note also that since $0 < b_n < 1$, we have $\log(1 - b_n) < 0$ for each n.

Now since $1 - b_n = a_{n+1}/a_n$ for every n, it is clear that $a_n = a_1(1 - b_1)(1 - b_2)(1 - b_3) \cdots (1 - b_{n-1})$ for each $n \geqslant 2$, and we therefore conclude that $a_n \to 0$ iff $\log a_n \to -\infty$ iff $\log a_1 + \Sigma_{i=1}^n \log(1 - b_i) \to -\infty$ iff $\Sigma \log(1 - b_n)$ diverges iff Σb_n diverges.

Returning now to the above example, we see that $b_n = 1/(2n + 2)$ for each n, and the obvious divergence of Σb_n implies that $a_n \to 0$. The same technique gives an easy proof of the convergence of such series as $\Sigma((-1)^n n^n/e^n n!)$, and the series $\Sigma\binom{\alpha}{n}$ of binomial coefficients with $\alpha > -1$.

Universal Topological Spaces

K. D. MAGILL, JR.
Department of Mathematics, SUNY, Buffalo, NY 14214

Let $U = \{a, b, c\}$ and let $\mathscr{T}_1 = \{U, \phi, \{a\}\}$. It has been known for a long time that U with the topology $\mathscr{T}_1$ is a *universal topological space* in the sense that any topological space whatsoever is homeomorphic to a subspace of some topological