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# A Graph Theoretic Summation of the Cubes of the First n Integers

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A Graph Theoretic Summation of the Cubes of the First $n$ Integers

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The complete graph $K_{n+1}$ contains $n+1$ vertices and $\binom{n+1}{2}$ edges. Iteratively building the complete graph $K_{n+1}$ by introducing vertices one at a time and counting the new edges incident to the new vertex provides a combinatorial proof that $\sum_{i=1}^{n} i = \binom{n+1}{2}$.

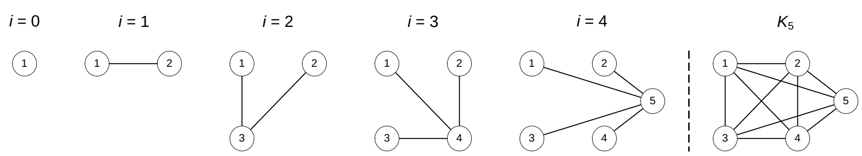
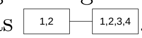



Figure 1: $\sum_{i=1}^{4} i = \binom{4+1}{2}$

Since $\sum_{i=1}^{n} i^3 = \binom{n+1}{2}^2$ it seems natural to look for a combinatorial proof that also uses graphs. Consider the complete bipartite graph $K_{\binom{n+1}{2},\binom{n+1}{2}}$ that contains $2\binom{n+1}{2}$ vertices and $\binom{n+1}{2}^2$ edges. As before, we will count the new edges incident to newly introduced vertices in $n$ stages. At stage $i$ we introduce $i$ new vertices to each side of the graph and count the edges incident to these new vertices. Since $\sum_{i=1}^{n} i = \binom{n+1}{2}$ this process enumerates all the edges in $K_{\binom{n+1}{2},\binom{n+1}{2}}$. New vertices on one side are adjacent only to vertices on the other side. When just considering the edges between the new vertices, the subgraph $K_{i,i}$ immediately appears with $i^2$ edges. It turns out that these $i^2$ edges along with the additional edges constructed between a new vertex on one side and an old vertex on the other side will always total $i^3$ new edges. This shows that $\sum_{i=1}^{n} i^3 = \binom{n+1}{2}^2$.

In order to see that we always introduce $i^3$ new edges at stage $i$, we will partition the new edges into complete bipartite graphs. At stage $i$, there exist

$\binom{i}{2} = \frac{i(i-1)}{2}$ previously introduced vertices on each side of the graph and the new vertices on each side are labeled $\binom{i}{2}+1, \binom{i}{2}+2..., \binom{i}{2}+i = \binom{i+1}{2}$. The partition of these edges into complete bipartite graphs depends upon the parity of $i$. Figure 2 illustrates these stages for $n = 5$. To prevent a deluge of edges in the graph, a complete bipartite graph such as $K_{2,4}$ is represented as 

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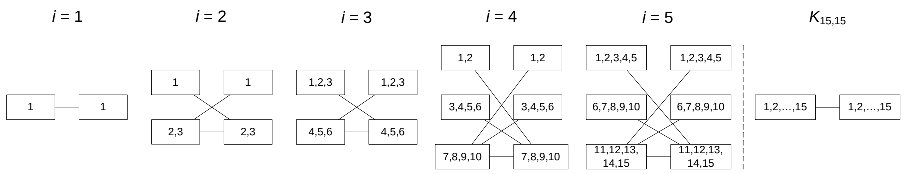


Figure 2: $\sum_{i=1}^{5} i^3 = \binom{5+1}{2}^2$

When $i$ is odd, the new edges quickly form $i$ disjoint copies of $K_{i,i}$. For odd $i$ we partition the old vertices into $\frac{i-1}{2}$ sets of $i$ vertices for each side. Both sets of $i$ new vertices are adjacent to each of the $\frac{i-1}{2}$ sets of $i$ vertices on the other side. This yields $2(\frac{i-1}{2}) = i-1$ additional copies of $K_{i,i}$. Along with the initial copy of $K_{i,i}$ on only the new vertices, we have $i$ copies of $K_{i,i}$ for a total of $i^3$ new edges.

When $i$ is even, we have to work a bit harder. For even $i$, we partition the old vertices on each side into $\frac{i}{2}-1$ sets of $i$ vertices and one set of $\frac{i}{2}$ vertices. This yields $2\left(\frac{i}{2}-1\right)$ copies of $K_{i,i}$ and two copies of $K_{\frac{i}{2},i}$ for $2\left(\frac{i}{2}-1\right)i^2+2\frac{i}{2}i = i^3-i^2$ edges. As before, with the original $K_{i,i}$ between the sets of new vertices, the total once again is $i^3$ new edges.

# References

[1] J. DeMaio and J. Tyson, Proof without words: a graph theoretic summation of the first $n$ integers, *The College Mathematics Journal* **38** (2007) 296.