2008

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Recommended Citation

Yang, B. (2008). Estimates of positive solutions for higher order right focal boundary value problem. Communications in Mathematical Analysis, 4(1), 1-9.

Communications in Mathematical Analysis
Volume 4, Number 1, pp. 1–9 (2008) *www.commun-math-anal.org*
ISSN 1938-9787

ESTIMATES OF POSITIVE SOLUTIONS FOR HIGHER ORDER RIGHT FOCAL BOUNDARY VALUE PROBLEM

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(Communicated by Toka Diagana)

Abstract

We consider the $(p,n-p)$ right focal boundary value problem. A new set of upper and lower estimates of positive solutions for the boundary value problem are obtained. These estimates implement and improve the ones in the literature.

AMS Subject Classification: 34B18.

Keywords: Focal boundary value problem; positive solution, upper and lower estimates.

1 Introduction

In this paper, we consider the $(p,n-p)$ right focal boundary value problem

$$u^{(n)}(t)+(-1)^{n-p+1}f(t,u(t))=0, \quad 0\le t\le 1, \tag{1.1}$$

$$\left.\begin{array}{ll} u^{(i)}(0)=0, & i=0,1,\cdots,p-1, \\ u^{(j)}(1)=0, & j=p,p+1,\cdots,n-1. \end{array}\right\} \tag{1.2}$$

Throughout this paper, we assume that

(H1) $f:[0,1]\times[0,\infty)\to[0,\infty)$ is a continuous function, n and p are fixed positive integers such that $1\le p\le n-1$.

The higher order right focal boundary value problem has been considered by many authors. For example, in 1991, Eloe and Henderson [8] studied the singular $(n-1,1)$ focal boundary value problem. In 1998, Henderson and Yin [11] considered the singular $(k,n-k)$ boundary value problem between conjugate and focal. The general $(p,n-p)$ focal boundary value

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problem was considered by Agarwal et al. in [3, 4, 5, 6, 7] and in Chapter 13 of [2]. If $n=2$ and $p=1$, then (1.1)-(1.2) reduces to the second order problem

$$u''(t)+f(t,u(t))=0, \quad 0\le t\le 1,$$
$$u(0)=u'(1)=0,$$

which has been considered by many authors, including Graef and Yang [10]. A special case of the problem (1.1)-(1.2) in which $n=4$ and $p=2$ is the model for the bending or deformation of a cantilever beam, and was considered by Yang [12] in 2005.

The Green's function $G(t,s)$ for the problem (1.1)-(1.2) is given by (see [6])

$$G(t,s)=\begin{cases} \dfrac{-1}{(n-1)!}\sum\limits_{i=0}^{p-1} C_{n-1}^{i} t^{i}(-s)^{n-i-1}, & t\ge s,\\ \dfrac{1}{(n-1)!}\sum\limits_{i=p}^{n-1} C_{n-1}^{i} t^{i}(-s)^{n-i-1}, & s\ge t. \end{cases}$$

It is known that $G(t,s)\ge 0$ for $(t,s)\in[0,1]\times[0,1]$ (see [1]). The problem (1.1)-(1.2) is equivalent to the integral equation

$$u(t)=\int_0^1 G(t,s)f(s,u(s))\,ds, \quad 0\le t\le 1. \tag{1.3}$$

For the problem (1.1)-(1.2), the following lower type estimate was proved by Agarwal and O'Regan in [5].

Theorem 1.1. *If $y\in C^n[0,1]$ satisfies* (1.2) *and $(-1)^{n-p}y^{(n)}(t)\ge 0$ for $0\le t\le 1$, then $y(t)\ge t^p y(1)$ for $0\le t\le 1$. In particular, if $y(t)$ is a positive solution to the problem* (1.1)-(1.2), *then $y(t)\ge t^p y(1)$ for $0\le t\le 1$.*

No upper estimate of positive solutions to the problem (1.1)-(1.2) has been obtained in the literature. The main purpose of this paper is to prove a new set of upper and lower estimates for positive solutions to the problem (1.1)-(1.2). Here by a positive solution, we mean a solution $u(t)$ such that $u(t)>0$ for $0<t<1$. The upper and lower estimates given in the next section implement and improve the one given by Theorem 1.1.

2 Estimates of Positive Solutions

We define the functions $a:[0,1]\to[0,1]$, $b:[0,1]\to[0,1]$, and $c:[0,1]\to[0,1]$ by

$$\begin{aligned} a(t) &= (n-1)\int_0^t (t-s)^{p-1}(1-s)^{n-p-1}\,ds,\\ b(t) &= t^{p-1},\\ c(t) &= n\int_0^t (t-s)^{p-1}(1-s)^{n-p}\,ds. \end{aligned}$$

The functions $a(t)$, $b(t)$, and $c(t)$ will be used to estimate positive solutions of the problem (1.1)-(1.2). It is easy to see that $a(0)=b(0)=c(0)=0$, $a(1)=b(1)=c(1)=1$, and that $a(t)$, $b(t)$, and $c(t)$ are continuous, nonnegative, and increasing on $[0,1]$. Below are the expressions for $a(t)$, $b(t)$, and $c(t)$ in two special cases:

(i) If $n = 2$ and $p = 1$, then $a(t) = t$, $b(t) = 1$, and $c(t) = 2t - t^2$.

(ii) If $n = 4$ and $p = 2$, then

$$a(t) = 3t^2/2 - t^3/2, \quad b(t) = t, \quad c(t) = 2t^2 - 4t^3/3 + t^4/3.$$

The next lemma is known [1].

Lemma 2.1. *If $u \in C^n[0,1]$ satisfies* (1.2)*, and*

$$(-1)^{n-p}u^{(n)}(t) \geq 0, \quad 0 \leq t \leq 1, \tag{2.1}$$

then $u'(t) \geq 0$ and $u(t) \geq 0$ for $0 \leq t \leq 1$.

The next two technical lemmas will be used to prove our new lower and upper estimates of the positive solutions to the problem (1.1)-(1.2).

Lemma 2.2. *Let $m \geq 1$ be a positive integer. If*

(A1) $u \in C^m[0,1]$ such that $u(0) = u'(0) = u''(0) = \cdots = u^{(m-1)}(0) = 0$, and

(A2) $u(t_0) = 0$ for some $t_0 \in (0,1]$,

then

(A3) there exist points $t_1, t_2, \cdots, t_m$ such that $t_0 > t_1 > t_2 > \cdots t_m > 0$ and

$$u^{(j)}(t_j) = 0 \quad \text{for} \quad j = 0,1,2,3,\cdots,m.$$

Proof. By mean value theorem, since $u(0) = u(t_0) = 0$, there exists $t_1 \in (0,t_0)$ such that $u'(t_1) = 0$. Since $u'(0) = u'(t_1) = 0$, there exists $t_2 \in (0,t_1)$ such that $u''(t_2) = 0$. In this way, (A3) can be proved easily by an induction on j. The proof is complete. □

Lemma 2.3. *Let $m \geq 1$ be a positive integer. Suppose that (A1), (A2), and (A3) hold. If*

$$u^{(m)}(t) \geq 0 \ \text{ on } \ (0,t_m), \ \ u^{(m)}(t) \leq 0 \ \text{ on } \ (t_m,1), \tag{2.2}$$

then for each $j = 0,1,2,\cdots,m-1$, we have

$$u^{(j)}(t) \geq 0 \ \text{ on } \ (0,t_j), \ \ u^{(j)}(t) \leq 0 \ \text{ on } \ (t_j,1). \tag{2.3}$$

Proof. Note that (2.2) means that $u^{(m-1)}(t)$ is nondecreasing on $[0,t_m]$ and nonincreasing on $[t_m,1]$. Since $u^{(m-1)}(0) = u^{(m-1)}(t_{m-1}) = 0$ and $0 < t_m < t_{m-1}$, we have

$$u^{(m-1)}(t) \geq 0 \ \text{ on } \ (0,t_{m-1}), \ \ u^{(m-1)}(t) \leq 0 \ \text{ on } \ (t_{m-1},1). \tag{2.4}$$

Now (2.4) implies that $u^{(m-2)}(t)$ is nondecreasing on $[0,t_{m-1}]$, and nonincreasing on $[t_{m-1},1]$. Since $u^{(m-2)}(0) = u^{(m-2)}(t_{m-2}) = 0$ and $0 < t_{m-1} < t_{m-2}$, we have

$$u^{(m-2)}(t) \geq 0 \ \text{ on } \ (0,t_{m-2}), \ \ u^{(m-2)}(t) \leq 0 \ \text{ on } \ (t_{m-2},1).$$

Hence, (2.3) is true for $j = m-1$ and $j = m-2$. Continuing the procedure, we can prove (2.3) by a (downward) induction on j. The proof is complete. □

Now we are ready to prove our new upper and lower estimates of positive solutions of the problem (1.1)-(1.2).

Lemma 2.4. *If* $u \in C^n[0,1]$ *satisfies* (1.2) *and* (2.1), *then*

$$u(t) \geq a(t)u(1), \quad 0 \leq t \leq 1. \tag{2.5}$$

Proof. Suppose $u \in C^n[0,1]$ satisfies (1.2) and (2.1). If we define

$$h(t) = u(t) - u(1)a(t), \quad 0 \leq t \leq 1, \tag{2.6}$$

then

$$(-1)^{n-p}h^{(n)}(t) = (-1)^{n-p}u^{(n)}(t), \quad 0 \leq t \leq 1. \tag{2.7}$$

To prove the lemma, it suffices to show that $h(t) \geq 0$ for $0 \leq t \leq 1$. Note that (2.6) implies that

$$h(0) = h'(0) = h''(0) = \cdots = h^{(p-1)}(0) = 0 \quad \text{and} \quad h(1) = 0. \tag{2.8}$$

If we let $t_0 = 1$, then $h(t_0) = 0$. By Lemma 2.2, there exist points $t_1, t_2, \cdots, t_p$ such that $t_0 > t_1 > t_2 > \cdots t_p > 0$ and

$$h^{(j)}(t_j) = 0 \quad \text{for} \quad j = 0, 1, 2, 3, \cdots, p. \tag{2.9}$$

In particular, $h^{(p)}(t_p) = 0$. At this point, there are two possible cases to consider:

Case I. If $p = n-1$, then (2.7) means that $h^{(p)}(t)$ is nonincreasing. Since $h^{(p)}(t_p) = 0$, we have

$$h^{(p)}(t) \geq 0 \text{ on } (0, t_p), \ \ h^{(p)}(t) \leq 0 \text{ on } (t_p, 1). \tag{2.10}$$

Case II. If $p < n-1$, then we see from (2.6) that $h^{(p)}(1) = h^{(p+1)}(1) = \cdots = h^{(n-2)}(1) = 0$. If we let $s_0 = 1 - t_p$, and define

$$v(t) = -h^{(p)}(1-t), \quad 0 \leq t \leq 1, \tag{2.11}$$

then

$$v(0) = v'(0) = \cdots = v^{(n-p-2)}(0) = 0 \quad \text{and} \quad v(s_0) = 0. \tag{2.12}$$

By Lemma 2.2, there exist points $s_1, s_2, \cdots, s_{n-p-1}$ such that $s_0 > s_1 > s_2 > \cdots > s_{n-p-1} > 0$ and

$$v^{(j)}(s_j) = 0 \quad \text{for} \quad j = 0, 1, 2, 3, \cdots, n-p-1. \tag{2.13}$$

In particular, $v^{(n-p-1)}(s_{n-p-1}) = 0$. We see from (2.7) and (2.11) that

$$v^{(n-p)}(t) = -(-1)^{n-p}h^{(n)}(1-t) \leq 0, \quad 0 \leq t \leq 1,$$

which implies that $v^{(n-p-1)}(t)$ is nonincreasing. Since $v^{(n-p-1)}(s_{n-p-1}) = 0$, we have

$$v^{(n-p-1)}(t) \geq 0 \text{ on } (0, s_{n-p-1}), \ \ v^{(n-p-1)}(t) \leq 0 \text{ on } (s_{n-p-1}, 1). \tag{2.14}$$

Since (2.12), (2.13), and (2.14) hold, by applying Lemma 2.3 to $v(t)$, we have for each $j = 0, 1, \cdots, n-p-1$, that

$$v^{(j)}(t) \geq 0 \text{ on } (0, s_j), \ \ v^{(j)}(t) \leq 0 \text{ on } (s_j, 1),$$

In particular, we have

$$v(t) \ge 0 \text{ on } (0, s_0), \;\; v(t) \le 0 \text{ on } (s_0, 1),$$

which implies (2.10).

Hence, in either case we have (2.10). Note that now (2.8), (2.9) and (2.10) all hold. Applying Lemma 2.3 again, this time to $h(t)$, we arrive at the fact that for each $j = 0, 1, \cdots, p$,

$$h^{(j)}(t) \ge 0 \text{ on } (0, t_j), \;\; h^{(j)}(t) \le 0 \text{ on } (t_j, 1).$$

In particular, we have

$$h(t) \ge 0 \text{ on } (0, t_0) = (0, 1), \;\; h(t) \le 0 \text{ on } (t_0, 1) = (1, 1) = \emptyset,$$

which simply means $h(t) \ge 0$ for $0 \le t \le 1$. The proof is complete. □

Lemma 2.5. *If $u \in C^n[0,1]$ satisfies* (1.2) *and* (2.1)*, then*

$$u(t) \le b(t)u(1) \quad \textit{for} \quad 0 \le t \le 1. \tag{2.15}$$

Proof. Suppose that $u \in C^n[0,1]$ satisfies (1.2) and (2.1). If we define

$$h(t) = b(t)u(1) - u(t), \quad 0 \le t \le 1, \tag{2.16}$$

then

$$(-1)^{n-p} h^{(n)}(t) = -(-1)^{n-p} u^{(n)}(t) \le 0, \quad 0 \le t \le 1. \tag{2.17}$$

We see from (2.16) that $h^{(j)}(1) = 0$ for $j = p, p+1, \cdots, n-1$. Therefore,

$$h^{(p)}(t) = (-1)^{n-p} \int_t^1 \frac{(s-t)^{n-p-1}}{(n-p-1)!} h^{(n)}(s)\, ds \le 0, \quad 0 \le t \le 1,$$

which means that $h^{(p-1)}(t)$ is nonincreasing on $[0,1]$. It is easy to see from (2.16) that $h(1) = 0$. At this point, there are two possible cases to consider:

Case I. If $p = 1$, then $h(t)$ is nonincreasing on $[0,1]$. Since $h(1) = 0$, we have $h(t) \ge 0$ on $[0,1]$.

Case II. If $p > 1$, then we see from (2.16) that

$$h(0) = h'(0) = \cdots = h^{(p-2)}(0) = 0, \text{ and } h(1) = 0. \tag{2.18}$$

If we let $t_0 = 1$, then $h(t_0) = 0$. By Lemma 2.2, there exist points $t_1, t_2, \cdots, t_{p-1}$ such that $t_0 > t_1 > t_2 > \cdots t_{p-1} > 0$ and

$$h^{(j)}(t_j) = 0 \quad \text{for} \quad j = 0, 1, 2, 3, \cdots, p-1. \tag{2.19}$$

In particular, we have $h^{(p-1)}(t_{p-1}) = 0$. Since $h^{(p-1)}(t)$ is nonincreasing on $[0,1]$, we have

$$h^{(p-1)}(t) \ge 0 \text{ on } (0, t_{p-1}), \;\; h^{(p-1)}(t) \le 0 \text{ on } (t_{p-1}, 1). \tag{2.20}$$

Since (2.18), (2.19), and (2.20) hold, we can apply Lemma 2.3 to $h(t)$ and arrive at the fact that $h(t) \ge 0$ on $[0,1]$.

Hence, it is true in either case that $h(t) \ge 0$ on $[0,1]$. The proof is complete. □

Lemma 2.6. *If $u \in C^n[0,1]$ satisfies* (1.2) *and* (2.1), *and $(-1)^{n-p}u^{(n)}(t)$ is nondecreasing on $[0,1]$, then*

$$u(t) \le c(t)u(1) \quad \textit{for} \quad 0 \le t \le 1.$$

Proof. If we define

$$h(t) = u(1)c(t) - u(t), \quad 0 \le t \le 1, \tag{2.21}$$

then

$$(-1)^{n-p}h^{(n)}(t) = L - (-1)^{n-p}u^{(n)}(t), \quad 0 \le t \le 1, \tag{2.22}$$

where $L := n(p-1)!(n-p)!(-1)^{n-p}u(1)$ is a constant. To prove the lemma, it suffices to show that $h(t) \ge 0$ for $0 \le t \le 1$. It is easy to see from (2.21) that

$$h(0) = h'(0) = \cdots = h^{(p-1)}(0) = 0, \text{ and } h(1) = 0. \tag{2.23}$$

If we let $t_0 = 1$, then $h(t_0) = 0$. By Lemma 2.2, there exist points $t_1, t_2, \cdots, t_p$ such that $t_0 > t_1 > t_2 > \cdots t_p > 0$ and

$$h^{(j)}(t_j) = 0 \quad \text{for} \quad j = 0, 1, 2, 3, \cdots, p. \tag{2.24}$$

In particular, we have $h^{(p)}(t_p) = 0$. We also see from (2.21) that $h^{(p)}(1) = h^{(p+1)}(1) = \cdots = h^{(n-1)}(1) = 0$. If we let $s_0 = 1 - t_p$, and define

$$v(t) = -h^{(p)}(1-t), \quad 0 \le t \le 1,$$

then

$$v(0) = v'(0) = \cdots = v^{(n-p-1)}(0) = 0, \text{ and } v(s_0) = 0. \tag{2.25}$$

By Lemma 2.2, there exist points $s_1, s_2, \cdots, s_{n-p}$ such that $s_0 > s_1 > s_2 > \cdots > s_{n-p} > 0$ and

$$v^{(j)}(s_j) = 0 \quad \text{for} \quad j = 0, 1, 2, 3, \cdots, n-p. \tag{2.26}$$

In particular, we have $v^{(n-p)}(s_{n-p}) = 0$. We see from (2.22) that

$$v^{(n-p)}(t) = -(-1)^{n-p}h^{(n)}(1-t) = -L + (-1)^{n-p}u^{(n)}(1-t), \quad 0 \le t \le 1.$$

Since $(-1)^{n-p}u^{(n)}(t)$ is nondecreasing on $[0,1]$, we have that $(-1)^{n-p}u^{(n)}(1-t)$ is nonincreasing in t. Therefore, $v^{(n-p)}(t)$ is nonincreasing in t. Since $v^{(n-p)}(s_{n-p}) = 0$, we have

$$v^{(n-p)}(t) \ge 0 \text{ on } (0, s_{n-p}), \;\; v^{(n-p)}(t) \le 0 \text{ on } (s_{n-p}, 1). \tag{2.27}$$

Note that (2.25), (2.26), and (2.27) all hold. By Lemma 2.3, we have

$$v(t) \ge 0 \text{ on } (0, s_0), \;\; v(t) \le 0 \text{ on } (s_0, 1),$$

which implies that

$$h^{(p)}(t) \ge 0 \text{ on } (0, t_p), \;\; h^{(p)}(t) \le 0 \text{ on } (t_p, 1). \tag{2.28}$$

Since (2.23), (2.24), and (2.28) hold, we can apply Lemma 2.3 to $h(t)$, and get that $h(t) \ge 0$ for $0 \le t \le 1$. The proof is complete. □

Theorem 2.7. *Suppose that (H1) holds. If $u \in C^n[0,1]$ is a positive solution of the problem* (1.1)-(1.2)*, then*

$$a(t)u(1) \le u(t) \le b(t)u(1) \quad \textit{for} \quad 0 \le t \le 1.$$

Proof. If $u \in C^n[0,1]$ is a positive solution of the problem (1.1)-(1.2), then $u(t)$ satisfies the boundary conditions (1.2) and the inequality (2.1). Now the theorem follows directly from Lemmas 2.4 and 2.5. The proof is complete. □

Theorem 2.8. *Suppose that (H1) and the following condition hold.*

(H2) $f(t,u)$ is nondecreasing in t and nondecreasing in u.

If $u \in C^n[0,1]$ is a positive solution of the problem (1.1)-(1.2)*, then*

$$a(t)u(1) \le u(t) \le c(t)u(1) \quad \textit{for} \quad 0 \le t \le 1.$$

Proof. If $u \in C^n[0,1]$ is a positive solution of the problem (1.1)-(1.2), then $u(t)$ satisfies the boundary conditions (1.2), and

$$(-1)^{n-p}u^{(n)}(t) = f(t,u(t)) \ge 0, \quad 0 \le t \le 1.$$

By Lemma 2.1, $u(t)$ is nondecreasing on $[0,1]$. By (H2), $(-1)^{n-p}u^{(n)}(t)$ is nondecreasing on the interval $[0,1]$. Now the theorem follows directly from Lemmas 2.4 and 2.6. The proof is complete. □

Lemma 2.9. *If $1 \le p \le n-1$, then $a(t) \ge t^p$ for $0 \le t \le 1$.*

Proof. If $p = n-1$, then $a(t) = t^p$ and the lemma is trivial. So we assume that $p < n-1$ in the remainder of the proof. If we define $h(t) = a(t) - t^p$, $0 \le t \le 1$, then

$$h(0) = h'(0) = \cdots = h^{(p-1)}(0) = h(1) = 0, \tag{2.29}$$

$$h^{(p)}(1) = -p!, \quad h^{(n)}(t) \equiv 0 \quad \text{on} \quad [0,1]. \tag{2.30}$$

To prove the lemma, it suffices to show that $h(t) \ge 0$ for $0 \le t \le 1$. If we let $t_0 = 1$, then $h(t_0) = 0$. By Lemma 2.2, there exist points $t_1, t_2, \cdots, t_p$ such that $t_0 > t_1 > t_2 > \cdots t_p > 0$ and

$$h^{(j)}(t_j) = 0 \quad \text{for} \quad j = 0,1,2,3,\cdots,p. \tag{2.31}$$

We take two cases to continue:

Case I: If $p = n-2$. In this case, since $h^{(p)}(t_p) = 0$ and (2.30) holds, we have

$$h^{(p)}(t) = -p! + \frac{p!}{1-t_p}(1-t), \quad 0 \le t \le 1.$$

Case II: If $p < n-2$. In this case, we have $h^{(p+1)}(1) = h^{(p+2)}(1) = \cdots = h^{(n-2)}(1) = 0$. Since $h^{(p)}(t_p) = 0$, $h^{(p)}(1) = -p!$, and $h^{(n)}(t) \equiv 0$ on $[0,1]$, we have

$$h^{(p)}(t) = -p! + \frac{p!}{(1-t_p)^{n-p-1}}(1-t)^{n-p-1}, \quad 0 \le t \le 1.$$

In either case, $h^{(p)}(t)$ is decreasing in t. Since $h^{(p)}(t_p) = 0$, we have

$$h^{(p)}(t) > 0 \text{ on } (0, t_p), \quad h^{(p)}(t) < 0 \text{ on } (t_p, 1). \tag{2.32}$$

Note that (2.29), (2.31), and (2.32) hold. If we apply Lemma 2.3 to $h(t)$, we get $h(t) \geq 0$ on $[0, 1]$. This completes the proof of the lemma. □

Remark 2.10. Lemma 2.9 shows that our lower estimate in Lemma 2.4 is better than the lower estimate in Theorem 1.1.

Remark 2.11. Note that it is possible that the problem (1.1)-(1.2) does not have a positive solution. Actually the upper and lower estimates given in Theorems 2.7 and 2.8 can help us find sufficient conditions for existence and nonexistence of positive solutions of the boundary value problem, which will be discussed in another paper. The reader is referred to [9, 12] for some works in this line.

Acknowledgment

The author thanks the anonymous referee for valuable comments.

References

[1] R. P. Agarwal, Focal boundary value problems for differential and difference equations, Mathematics and its Applications, 436, Kluwer Academic Publishers, Dordrecht, 1998.

[2] R. P. Agarwal, D. O'Regan and P. J. Y. Wong, Positive solutions of differential, difference and integral equations, Kluwer Academic Publishers, Dordrecht, 1999.

[3] R. P. Agarwal and D. O'Regan, Superlinear higher order boundary value problems, *Aequationes Math.* **57**:2-3 (1999), pp 233-240.

[4] R. P. Agarwal and D. O'Regan, Twin solutions to singular boundary value problems, *Proc. Amer. Math. Soc.* **128**:7 (2000), pp 2085-2094.

[5] R. P. Agarwal and D. O'Regan, Right focal singular boundary value problems, *ZAMM Z. Angew. Math. Mech.* **79**:6 (1999), pp 363-373.

[6] P. R. Agarwal, D. O'Regan and V. Lakshmikantham, A note on multiple solutions for $(p, n-p)$ focal and (n, p) problems, *Nonlinear Stud.* **7**:1 (2000), pp 18-24.

[7] P. R. Agarwal, D. O'Regan and V. Lakshmikantham, Singular $(p, n-p)$ focal and (n, p) higher order boundary value problems, *Nonlinear Anal.* **42**:2 (2000), pp 215-228.

[8] P. W. Eloe and J. Henderson, Singular nonlinear boundary value problems for higher order ordinary differential equations, *Nonlinear Anal.* **17**:1 (1991), pp 1-10.

[9] J. R. Graef and B. Yang, Existence and nonexistence of positive solutions of fourth order nonlinear boundary value problems, *Appl. Anal.* **74**:1-2 (2000), pp 201-214.

[10] J. R. Graef and B. Yang, Boundary value problems for second order nonlinear differential equations, *Commun. Appl. Anal.* **6**:2 (2002), pp 273-288.

[11] J. Henderson and W. Yin, Singular $(k,n-k)$ boundary value problems between conjugate and right focal, Positive solutions of nonlinear problems, *J. Comput. Appl. Math.* **88**:1 (1998), pp 57-69.

[12] B. Yang, Positive solutions for a fourth order boundary value problem, *Electron. J. Qual. Theory Differ. Equ.* **2005**:3 (2005), pp 1-17.