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The Slope Mean and Its Invariance Properties

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Euclidean area measure were well-defined in the noneuclidean plane we would have

$$\text{in} \quad \triangle\, UVW, \qquad \frac{uh_u}{2} = \frac{vh_v}{2},$$

$$\text{and in} \quad \triangle\, UMW, \qquad \frac{(u/2)h_u}{2} = \frac{vh_m}{2},$$

and so $h_v = 2h_m$. However, we should have $h_v > 2h_m$, as we showed before in FIGURE 1, which means that at least one of the above two equations is false. Hence, Euclidean area measure is not applicable in noneuclidean geometry.

How does one prove that (base $\times$ altitude)$/2$ is well-defined in Euclidean geometry, and what accounts for the difference in noneuclidean geometry? As is often the case in Euclidean geometry, one verifies the equation between two products of segments by transforming it into one between two quotients and then applies a proportionality theorem [**3**, § 20]. And that is exactly what does not work in noneuclidean geometry; in FIGURE 1

$$\frac{B'A'}{OA'} > \frac{BA}{OA}.$$

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The Slope Mean and Its Invariance Properties

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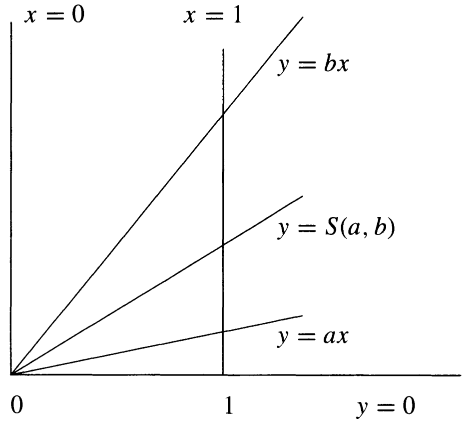
For $a, b > 0$, we know that the arithmetic mean $A(a, b) = (a + b)/2$ produces the midpoint of the segment $[a, b]$ on the real line. But what if we interpret a and b as slopes? A more natural mean in this context could be the "intermediate" slope, specifically, the positive slope $S(a, b)$ of the line $y = S(a, b)x$ that bisects the angle formed by the lines $y = ax$ and $y = bx$. As $a, b > 0$ vary in the figure, one senses that $S(a, b)$ is different from $A(a, b)$, but nonetheless has characteristics often associated with a mean.



Originally we chanced upon this mean while statistically comparing various linear regression methods [**9**]. We randomly perturbed a set of points on a line of slope m, repeating many times. For a particular method, we computed an average slope $\hat{m}$ to compare with the underlying slope m. Once in a while a random sample of perturbed points produced a near vertical regression line; this was a problem: since we used the arithmetic mean to compute $\hat{m}$, the corresponding near-infinite slope would not be canceled by the near-zero slope of a near horizontal line. We felt that it would be more meaningful to compute $\hat{m}$ by identifying slopes to angles, which led us to consider the mean

$$S(x_1, x_2, \ldots, x_n) = \tan\big((\tan^{-1} x_1 + \tan^{-1} x_2 + \cdots + \tan^{-1} x_n)/n\big). \qquad (1)$$

Before proceeding, we must carefully consider what we mean by *mean*. Typically a mean M is a function from $(0, \infty) \times (0, \infty) \times \cdots \times (0, \infty)$ to $(0, \infty)$ satisfying $\min\{x_1, x_2, \ldots, x_n\} \leq M(x_1, x_2, \ldots, x_n) \leq \max\{x_1, x_2, \ldots, x_n\}$ (intermediacy) and with an output value independent of the arrangement of input values (symmetry).

The classic means—the arithmetic, geometric, and harmonic—are defined respectively by

$$\begin{aligned} A(x_1, x_2, \ldots, x_n) &= (x_1 + x_2 + \cdots + x_n)/n, \\ G(x_1, x_2, \ldots, x_n) &= \sqrt[n]{x_1 x_2 \cdots x_n}, \\ H(x_1, x_2, \ldots, x_n) &= n/(1/x_1 + 1/x_2 + \cdots + 1/x_n). \end{aligned}$$

It is easy to see that these, as well as (1), are means as defined above. Many other means and families of means can be found in the vast literature on means [**1, 3, 6, 7, 10**].

Depending on focus and context, there is some variation in definition of *mean*. For example, sometimes continuity is assumed and other times symmetry is not required. We contend that often the definition is restricted to positive numbers for convenience and out of geometric tradition [**4**]. This requirement avoids problems in, for example, G and H, but is arguably unnaturally restrictive for A.

Define $D = \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n > 0 \text{ or } x_1, x_2, \ldots, x_n < 0\}$, and, for the purposes of this note, define a *mean* to be a function $M : D \to \mathbb{R}$ satisfying intermediacy and symmetry and

$$M(x_1, x_2, \ldots, x_n) = -M(-x_1, -x_2, \ldots, -x_n) \quad \text{for} \quad x_1, x_2, \ldots, x_n < 0. \qquad (2)$$

If necessary, we extend a mean on $(0, \infty) \times (0, \infty) \times \cdots \times (0, \infty)$ to D by requiring (2).

Returning to the definition of S given by (1), we see that $S(x_1, x_2)$ returns the slope of the line that bisects the angle formed by lines with slopes x_1 and x_2. Because of this interpretation, we call S the *slope mean*, though the "slope to arc back to slope mean" would be a more apt description. In the literature, a special focus has been placed on homogeneous means. Although the slope mean is not homogeneous, we will see how it is closely related to the three classic means.

The notion of invariance Much attention is given to means M that are *homogeneous*, meaning that $M(\xi x_1, \dots, \xi x_n) = \xi M(x_1, \dots, x_n)$, for $\xi > 0$. We generalize this notion, and say that a mean M on D is *invariant* under a real valued function f if $M(f(x_1), f(x_2), \dots, f(x_n)) = f(M(x_1, x_2, \dots, x_n))$ for all $(x_1, x_2, \dots, x_n) \in D$ for which both sides of the equality are defined. Thus, we will also call a homogeneous mean *scalar invariant*, due to the fact that it is invariant under $f_\xi(x) = \xi x$ for all $\xi \neq 0$.

It is straightforward to verify that the arithmetic mean is scalar invariant and invariant under translation given by $g_\tau(x) = x - \tau$, $\tau \in \mathbb{R}$. Moving on, it is also easily seen that the geometric mean is also scalar invariant. Moreover, it is invariant under reciprocation $r(x) = 1/x$. Other well-known means (see Eves' list [**4**, p. 200]) do not have this invariance, but the slope mean shares such a property with the geometric mean, as we will now show:

Fix $x_1, x_2, \dots, x_n > 0$ and choose $\theta_1, \theta_2, \dots, \theta_n \in (0, \pi/2)$ such that $\tan\theta_i = x_i$ for all $i = 1, 2, \dots, n$. Then $\tan^{-1}(1/x_i) = \pi/2 - \theta_i$ for each i, which, together with the relations between $\tan x$ and $\cot x$, leads to

$$\begin{aligned} S(r(x_1), \dots, r(x_n)) &= \tan\Big(\big(\tan^{-1}(1/x_1) + \cdots + \tan^{-1}(1/x_n)\big)/n\Big) \\ &= \cot\left(\left(\sum_{i=1}^{n} \tan^{-1} x_i\right) \Big/ n\right) = r\big(S(x_1, x_2, \dots, x_n)\big). \end{aligned}$$

We leave to the reader to check that the harmonic mean is scalar invariant but the slope mean is not. The relation between the arithmetic and harmonic means

$$A\big(r(x_1), r(x_2), \dots, r(x_n)\big) = r\big(H(x_1, x_2, \dots, x_n)\big) \tag{3}$$

will be used in the remainder of the note.

Comparison with the classic means The three classic means given in the introduction can be compared by the most frequently proven inequalities of classical analysis [**2, 5**]:

$$H(x_1, x_2, \dots, x_n) \le G(x_1, x_2, \dots, x_n) \le A(x_1, x_2, \dots, x_n) \quad \text{for all} \quad x_i > 0. \tag{4}$$

The second inequality in (4) is the celebrated *Geometric-Arithmetic Mean Inequality*. The slope mean, like the geometric mean, is also trapped between A and H.

THEOREM 1. *For all $x_1, x_2, \cdots, x_n > 0$, we have*

$$H(x_1, x_2, \dots, x_n) \le S(x_1, x_2, \dots, x_n) \le A(x_1, x_2, \dots, x_n).$$

Proof. The second inequality follows from the fact that $f(x) = \tan^{-1} x$ is concave down for $x > 0$. The first inequality follows from the invariance of S under $r(x) = 1/x$ and (3). ∎

The slope mean S is not scalar invariant, which in turn implies that a nontrivial family of means S_ξ can be introduced by

$$S_\xi(x_1, x_2, \dots, x_n) = (1/\xi)S(\xi x_1, \xi x_2, \dots, \xi x_n), \quad \text{for all} \quad \xi > 0. \tag{5}$$

The following result will further extend the assertion in Theorem 1 and connect the slope mean to the arithmetic and harmonic means.

THEOREM 2. *Let S_ξ be defined by* (5). *Then, for all* $x_1, x_2, \cdots, x_n > 0$,

(a) $H(x_1, x_2, \dots, x_n) \le S_\xi(x_1, x_2, \dots, x_n) \le A(x_1, x_2, \dots, x_n)$ *for all* $\xi > 0$.

(b) $\lim_{\xi \to 0^+} S_\xi = A$ *and* $\lim_{\xi \to \infty} S_\xi = H$.

Proof. Part (a) follows from Theorem 1 and the fact that A and H are scalar invariant. Part (b) is easily verified by L'Hospital's rule. ∎

Throughout the rest of the note we will concentrate on means with $n = 2$. Let $a, b > 0$, thinking of a and b as slopes. In this case, (1) simplifies to give the mean $m = S(a, b)$ of a and b as $m = \tan((\tan^{-1} a + \tan^{-1} b)/2)$. Therefore, m satisfies $\tan(2\tan^{-1} m) = \tan(\tan^{-1} a + \tan^{-1} b)$. Applying the angle sum identity for tangent we end up with a quadratic equation in m whose positive root can be expressed as

$$m = S(a, b) = \left(ab - 1 + \sqrt{(a^2 + 1)(b^2 + 1)}\right) / (a + b), \tag{6}$$

under the assumption that $ab \ne 1$. If $ab = 1$, then (6) yields $S(a, b) = S(a, 1/a) = 1$, which is consistent with (1) and the geometric interpretation of S. Therefore, for all $a, b > 0$ (and in fact for all $(a, b) \in D$) the formula (6) provides the slope of the line that bisects the angle formed by two lines of slopes a and b.

Characterization by invariance The classic means A, G, and H are scalar invariant. Mathematically, it is interesting to determine the class of functions under which a given mean is invariant; we will also see that a mean is uniquely determined by the class of functions under which it is invariant. We will study characterizations by invariance for arithmetic, geometric, harmonic, and slope means.

As pointed out earlier, A is invariant under all the functions

$$f_\xi(x) = \xi x, \quad \xi \ne 0 \quad \text{and} \quad g_\tau(x) = x - \tau, \ \tau \in \mathbb{R}. \tag{7}$$

More importantly, A is determined uniquely by these two sets of invariances as follows.

We assume that M is any mean on D invariant under the functions given by (7). Fix $(a, b) \in D$, and let $m = M(a, b)$. Using symmetry and scalar invariance, we have $m = M(b, a) = -M(-b, -a)$. On the other hand, the translation invariance gives $M(-b, -a) = M(a - (a + b), b - (a + b)) = M(a, b) - (a + b) = m - (a + b)$. Therefore $2m = a + b$ or $M(a, b) = A(a, b)$. Thus, the arithmetic mean is the only mean invariant under scaling and translation.

Next, we turn our attention to the geometric mean G. It turns out that the invariances under $f_\xi(x) = \xi x, \xi \ne 0$ and $r(x) = 1/x$ are enough to determine G. We encourage the reader to prove this result by adapting the argument for the arithmetic mean, or otherwise.

Moving on, it is not hard to verify that H is invariant under $f_\xi(x) = \xi x, \xi \ne 0$ and $g_\tau(x) = x/(1 - \tau x), \tau \in \mathbb{R}$. (We used the relation (3) to find the second invariance family.) The proof of Theorem 3 below suggests a method to show that these two families of invariances in fact determine H.

To finish, we now turn our attention to the slope mean. Since we have already shown that S is invariant under $r(x) = 1/x$, we know that S cannot be invariant under scaling, otherwise $S = G$. Is there a second invariance that determines S?

Now is a good time to think geometrically. Let $a, b > 0$ and consider the three lines through the origin and containing the points $A(1, a)$, $B(1, b)$, and $C(1, S(a, b))$. It is easily seen that $1/a$, $1/b$ and $1/S(a, b)$, respectively, are the slopes of the same lines, but taken with respect to the y-axis. The geometric meaning of the slope mean indicates $S(1/a, 1/b)$ is also the slope of the line containing C with respect to the y-axis. Hence, invariance under r, $S(1/a, 1/b) = 1/S(a, b)$, is now geometrically obvious.

Continuing to think this way, we find another natural invariance for S: rotation by a fixed angle. After a little work, we have that S is invariant under

$$f_\rho(x) = (x + \rho)/(1 - \rho x),\ \rho \in \mathbb{R} \quad \text{and} \quad r(x) = 1/x. \tag{8}$$

Moreover, S is determined by these two invariances.

THEOREM 3. *If a mean $M : D \to \mathbb{R}$ is invariant under all the functions in* (8), *then $M = S$ on D.*

Proof. Assume $M : D \to \mathbb{R}$ is invariant under the functions in (8). Fix $a, b > 0$ and let $m = M(a, b)$. The key observation is that the system of equations $f_\rho(a) = r(b)$ and $f_\rho(b) = r(a)$ admits a solution, namely $\rho = (1 - ab)/(a + b)$. Using symmetry and the invariances r and f_ρ (where $\rho = (1 - ab)/(a + b)$), we have

$$m = M(a, b) = M(b, a) = \frac{1}{M(1/b, 1/a)} = \frac{1}{M(f_\rho(a), f_\rho(b))} = \frac{1}{f_\rho(m)},$$

where the last equality is valid provided that $f_\rho(m)$ is defined. But if $m = 1/\rho$, then the intermediacy of $m = M(a, b)$ leads either to $b^2 \le -1$ (when $a < b$) or $a^2 \le -1$, both of which are impossible.

Thus, $m = 1/f_\rho(m)$ or $m = -\rho \pm \sqrt{\rho^2 + 1}$. Applying intermediacy one more time and substituting for ρ yields

$$m = \frac{ab - 1 + \sqrt{(ab - 1)^2 + (a + b)^2}}{a + b}.$$

Thus, $m = S(a, b)$ given by (6). ∎

Note that the technique employed in Theorem 3 can also be used to prove those previously mentioned characteristic results through invariance for arithmetic, geometric, and harmonic means. Also note that the families of invariances are algebraic subgroups under composition of the group of fractional linear transformations (provided functions are considered equal if they differ at a finite number of points).

Further notes We have focused on quasi-arithmetic means, means of the form $f^{-1}((f(x_1) + f(x_2) + \cdots + f(x_n))/n)$. Whenever f is a monotone function defined on $(0, \infty)$, this generates a mean. In particular, taking $f(x)$ to be x, $\ln x$, $1/x$, and $\tan^{-1} x$ generates A, G, H, and S, respectively. It should be noted that some of the general theory of quasi-arithmetic means, developed in Hardy, Littlewood, and Pólya's *Inequalities* [**5**], can be applied to obtain and extend the comparison results above.

Besides the quasi-arithmetic means, there are other families of means containing the slope mean. A beautifully simple class of means, going back to 1933 [**8**], is generated by any function $f : (0, \infty) \to (0, \infty)$ as $(f(a)b + f(b)a)/(f(a) + f(b))$. Taking $f(x)$ to be 1, $\sqrt{x}$, and x generates A, G, and H, respectively. Here the slope mean is generated by $f(x) = \sqrt{x^2 + 1}$. More recently, Dietel and Gordon [**3**] generated means from functions $f : (0, \infty) \to (0, \infty)$ and their tangent lines, which is a special case of the means in Horwitz [**6**]. Under the assumption that $f''(x)$ is nonzero and continuous, a mean is given by the x-coordinate of the intersection of the tangent lines to $y = f(x)$ at $x = a$ and $x = b$. In this case, taking $f(x)$ to be x^2, $\sqrt{x}$, and $1/x$ generates A, G, and H, respectively. The slope mean also belongs to this family, generated by $f(x) = \sqrt{x^2+1}$.

Some families of means do not contain the slope mean because the means are homogeneous, and yet there are still other families [**10**] where it is not clear whether or not the slope mean is a member.

In their study of the three classic means, Bullen, Mitrinović, and Vasić implicitly characterized these means through a family of functions as follows. Let $\{f_\lambda(x) : \lambda \in \mathbb{R}\}$ be a family of functions indexed by λ such that $f_\lambda^{-1}(x) = f_\lambda(x)$ and suppose that for every pair $(a, b) \in (0, \infty) \times (0, \infty)$, there exists a unique index $\lambda \equiv \lambda(a, b)$ such that $f_\lambda(a) = b$. Then $m = M(a, b)$ can be defined by $f_\lambda(m) = m$. It can be seen that our characterization of means by two sets of functions leads to such a characterization using a single family of functions.

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A Carpenter's Rule of Thumb

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In an episode of the PBS television series "The New Yankee Workshop," host and master carpenter Norm Abram needed to construct a rectangular wooden frame as part of a piece of furniture he was building. After gluing and clamping four pieces of wood together to form a rectangle, he checked the rectangle for squareness by measuring the two diagonals to determine whether or not they were of equal length. Upon finding a small difference in the two measurements, he announced that he would "split the difference." He proceeded to carefully nudge the top corner of the frame at the end of the longer diagonal until his measuring tape indicated that its length was the average of his two original diagonal measurements. He then said he was satisfied that the frame