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Asymptotic Properties of Equilibrium in Discriminatory and Uniform Price IPV Multi-Unit Auctions

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Abstract

This paper confronts the tractability problems that accompany IPV auction models with multi-unit bidder demands. Utilizing a first order approach, the asymptotic properties of symmetric equilibria in discriminatory and uniform price auctions are derived. It is shown that as the number of bidders increases, equilibrium bids converge to valuations in both discriminatory auctions and uniform price auctions where the price paid is determined by the lowest winning bid, thus indicating that the limiting case of these auctions correspond to price taking as in neoclassical models of consumer behavior. However, when the uniform price paid is tied to the highest losing bid, price taking behavior does not ensue and ex post inefficient allocations ares possible. The impact of our results on analysis of k-double auctions with multi-unit bidders demands is also discussed.

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1. Introduction

The majority of results established by the literature on auctions with multi-unit bidder demands have been limited due to an absence of analytic solutions (see Ausubel and Cramton 2002, Black and DeMeza 1992, Engelbrecht-Wiggans and Kahn 1998a, 1998b, Katzman 1995, 1999, and Noussair 1995). The inherent problem in examining auctions with multi-unit demands stems from the complexity of the first order conditions in the bidder's problem. This complexity results from the fact that bidders submit multiple bids, each potentially formulated using a different bid function. That is, the amount that a bidder's kth bid is shaded below its corresponding valuation may differ from the amount that his $k+1$st bid would be shaded below the same valuation, a strategy that we refer to as *differential shading.*

Engelbrecht-Wiggans (1999), Swinkels (1999, 2001), Jackson and Kremer (2002), and Chakraborty and Engelbrecht-Wiggans (2005) have avoided tractability problems by examining efficiency and prices asymptotically. For instance, Swinkels (1999, 2001) shows that both the discriminatory and uniform price auctions generate ex ante efficiency when there are a large number of bidders. His work is appealing in that no assumptions regarding symmetry of bidders, nor equilibrium are necessary to generate his results. However, while he offers valuable insight into equilibrium behavior in the limit, this approach admittedly is not "able to draw the connection between limit equilibria and the standard equilibrium of simple auctions." Our paper characterizes just that link by deriving the asymptotic properties of equilibrium directly from an individual bidder's maximization problem.

Our paper is less concerned with the asymptotic efficiency of auctions than with whether models of those auctions generate price taking behavior as the number of bidders grows large. That is, we ask if the limiting case of these auctions is in line with the neoclassical assumptions of price taking. We find that price taking does result in the discriminatory auction and uniform auction where the price paid equals the lowest winning bid. However, whenever the price paid in a uniform price auction is tied to the highest losing bid,[1] differential shading persists and price taking does not emerge.

Finally, an issue related to bid shading in uniform price auctions is bid/offer misrepresentation in the k-double auction studied by Rustichini *et al.* (1994)

[1] This uniform price auction has been examined quite often given its similarity to a second price single object auction where the highest losing bid determines the price paid.

where buyers (sellers) demand (supply) a single unit. They show that price taking emerges as the number of bidders grows. We develop a k-uniform price auction similar to their k-double auction and our results indicate that bid misrepresentation will not disappear in k-double auctions if multiple units are demanded by individual bidders and that the neoclassical model of markets is not the limiting case of the k-double auction.

2. The Environment

Discriminatory and uniform price auctions are used to sell m homogeneous objects to $N+1$ risk neutral bidders. Each bidder $(n = 1, \ldots, N+1)$ has a positive valuation for $d \leq m$ objects. The marginal valuations of bidder n are denoted $v_n(h)$, $h = 1, \ldots, d$. A downward sloping demand property is imposed on each bidder's valuations in that, $v_n(1) \geq v_n(2) \geq \cdots \geq v_n(d)$. Bidder's valuation vectors, $\mathbf{v}$, are drawn independently from a joint distribution function $F(\mathbf{v})$ that is continuously differentiable in all of its arguments with an atomless corresponding distribution function, the zero vector as a lower bound, and a finite symmetric vector as an upper bound.

In order to formulate a bidder's maximization problem in each auction, that bidder's probabilistic beliefs about his opponents' bids must be specified. This section specifies the distributions of these bids in general terms. In each auction, individual bidders submit d bids. In a given auction, an individual bidder competes against the highest m of the dN bids placed by his opponents. Let $\Delta_k(b)$ denote the cumulative distribution of the kth highest of these bids. Clearly, these distributions depend on the strategies employed by the bidder's opponents, which themselves will depend on the auction format.

The ex ante symmetry of bidders leads to focus on symmetric equilibria. Assume that in auction form a (= discriminatory, uniform) each bidder believes that his opponents are using the bid function, $B_a(\mathbf{v}) : \mathbf{R}^d \rightarrow \mathbf{R}^d$, that is continuously differentiable and monotonic increasing in all of its arguments. The fact that opponents' bid strategies and valuation distributions are identical dictates that the probability measure over any one opponent's bids is identical to that over any other opponent's bids. Let $\alpha_j(b)$, $j = 0, \ldots, d$, represent the probability that a bid b, is less than exactly j of another bidder's bids and higher than the other $d - j$ of that other bidder's bids. By the above assumptions $\alpha_j'(b)$ may be positive or negative with the exception that $\alpha_0^{'}(b) > 0 \; \forall b$. Letting i denote the number of bids another player has above a bid (b), λ_i indicates the number of individual bidders for which this is true. Using this formulation, $\Delta_k(b)$ is defined by the following recurrence relation, with $\Delta_0(b) = 0$,

$$\Delta_k(b) = \Delta_{k-1}(b) + \sum_{\sum i\lambda_i = k-1} \left\{ \binom{N}{\lambda_1, ..., \lambda_d} \prod_{w=0}^{d} \alpha_w(b)^{\lambda_w} \right\}, k = 1, \ldots, dN \quad (1)$$

where $\lambda_0 = N - \sum_{j=1}^{d} \lambda_j$, $\binom{N}{\lambda_1,...,\lambda_d} = \binom{N}{\lambda_1}\binom{N-\lambda_1}{\lambda_2} \cdots \binom{N-\eta}{\lambda_d}$, and $\eta = \sum_{i=1}^{d-1} \lambda_i$.[2] It follows that the density of the kth highest bid is defined by the following recurrence relation, with $\delta_0(b) = 0$,

$$\delta_k(b) = \delta_{k-1}(b) + \sum_{l=0}^{d} \sum_{\sum i\lambda_i = k-1} \left\{ \binom{N}{\lambda_1, ..., \lambda_d} \frac{\alpha_l'(b)\lambda_l}{\alpha_l(b)} \prod_{w=0}^{d} \alpha_w(b)^{\lambda_w} \right\}, k = 1, \ldots, dN. \quad (2)$$

Before continuing, an example of how λ works will ease the burden on the reader. $\Delta_3(b) - \Delta_2(b)$ is the probability that b is less than exactly three opponents' bids and greater than the remaining $dN - 3$ bids. The three ways that this can happen are (1) b is less than a single bid from three different opponents in which case $\lambda_0 = N - 3, \lambda_1 = 3$, and $\lambda_2 = \lambda_3 = \cdots = \lambda_d = 0$. (2) b is less than one of a single opponent's bids, less than two of another opponent's bids, and greater than all other opponents' bids in which case $\lambda_0 = N - 2, \lambda_1 = 1, \lambda_2 = 1$, and $\lambda_3 = \lambda_4 = \cdots = \lambda_d = 0$. (3) b is less than three of a single opponent's bids and greater than all other bids in which case $\lambda_0 = N - 1, \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 1$, and $\lambda_4 = \lambda_5 = \cdots = \lambda_d = 0$. Notice that in each case $\sum_{i=1}^{d} i\lambda_i = 3$ and that $\sum_{j=0}^{d} \lambda_j = N$ as is required.

3. The Discriminatory Auction

A bidder that wins $t \leq d$ objects pays $\sum_{i=1}^{t} b_i$. The bidder's problem is

$$\begin{aligned} \max_{b_1,\ldots,b_d} \quad & \sum_{j=1}^{d} (v(j) - b_j)\Delta_{m+1-j}(b_j) \\ \text{s.t.} \quad & b_n \geq b_l, \quad 1 \leq n < l \leq d. \end{aligned} \quad (3)$$

[2] The cobinatorial term in Eq. (1) is similar to the combinatorial expression of the multinomial expansion, $(a+b)^{k-1}$. The primary difference is that the multinomial expansion is summed over the values of λ such that $\sum_{i=1}^{k-1} \lambda_i = k - 1$, whereas our sum is taken over values of λ for which $\sum_{i=1}^{k-1} i\lambda_i = k - 1$.

If the constraints do not bind, the relevant first order conditions are

$$(v(j) - b_j)\delta_{m+1-j}(b_j) - \Delta_{m+1-j}(b_j) = 0, \quad j = 1, \ldots, d. \tag{4}$$

However, the constraints will bind in this case for certain realizations of valuations. Engelbrecht-Wiggans and Kahn (1998b) and Katzman (1995) first showed the intuition behind the binding constraints using the case where two units are sold. In that case, the system of Eqs. (4) reduces to

$$\begin{aligned} (v(1) - b_1)\delta_2(b_1) - \Delta_2(b_1) &= 0 \\ (v(2) - b_2)\delta_1(b_2) - \Delta_1(b_2) &= 0, \end{aligned}$$

the solution of which is not analytically tractable and can only be obtained using numerical techniques. The lack of analytic tractability in a system of equations of this type is not new to auction theory. Maskin and Riley (2000) have encountered a similar system of equations when analyzing single object auctions in the presence of ex ante bidder asymmetry. Perhaps surprisingly, the intuition provided by Maskin and Riley (2000) applies here as well, despite the fact that bidders are ex ante symmetric. This is most easily seen in the case of two objects being auctioned. In such a situation, a bidder's high bid must only be greater than the second highest of his opponents' bids while his low bid must exceed all of his opponents' bids. The result is that the bidder formulates his low bid using a more aggressive function than that used to formulate his high bid. This difference in aggressiveness creates a tendency for low bids to be above high bids when the two valuations are too "close" together. When this occurs, the constraint is violated and Kuhn-Tucker optimization is needed. This creates an additional equation that can be used to eliminate the Lagrangian multiplier. The result is a third equation in the system,

$$(\overline{L} - b^*)\delta_2(b^*) - \Delta_2(b^*) + (\overline{L} - b^*)\delta_1(b^*) - \Delta_1(b^*) = 0 \tag{5}$$

where $\overline{L}$ is defined as the highest low valuation that can lead to identical bids of b^*. Figure 1 shows the equilibrium level curves. There are two regions; one where different bids, b_1 and b_2, are submitted (when valuations are far enough apart), the other where identical bids, b^*, are submitted (when valuations are close enough together). Differential shading is seen by the fact that the level curves for the high and low bids do not meet at the 45^o line.

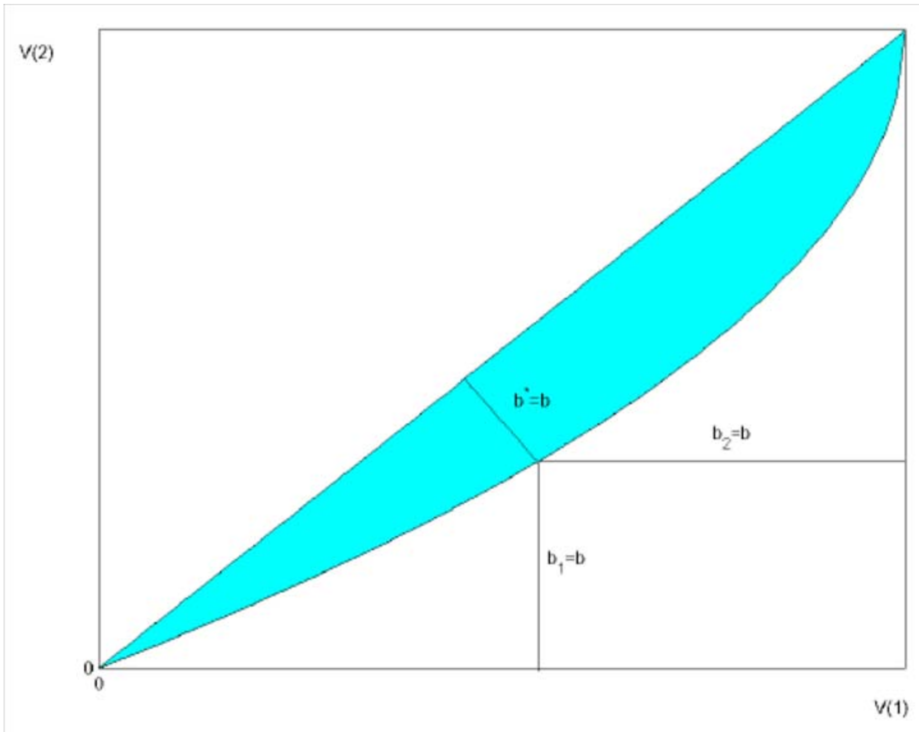

Figure 1: Discriminatory Auction Level Curves

4. The Uniform Price Auction

This section formulates a bidder's problem for a k-uniform price auction where the price paid by winning bidders is k times the highest losing bid plus $(1-k)$ times the lowest winning bid. A bidder that wins $t \leq d$ objects in this auction pays t times the market clearing price. The bidder's problem is

$$\max_{b_1,\ldots,b_d} \sum_{j=1}^{d} v(j)\Delta_{(m+1-j)}(b_j)$$
$$-k\left[\sum_{j=1}^{d}\left\{j\int_{b_{j+1}}^{b_j} x\delta_{(m+1-j)}(x)dx\right\} + \sum_{j=1}^{d}(j-1)b_j[\Delta_{m+2-j}(b_j) - \Delta_{m+1-j}(b_j)]\right]$$
$$-(1-k)\left[\sum_{j=1}^{d}\left\{j\int_{b_{j+1}}^{b_j} x\delta_{(m-j)}(x)dx\right\} + \sum_{j=1}^{d} jb_j[\Delta_{m+1-j}(b_j) - \Delta_{m-j}(b_j)]\right]$$
$$\text{s.t. } b_n \geq b_l \ \forall\ 1 \leq n < l \leq d, \quad b_{d+1} = 0.$$

The resulting first order conditions are

$$[v(j) - b_j]\delta_{m+1-j}(b_j) - k(j-1)[\Delta_{m+2-j}(b_j) - \Delta_{m+1-j}(b_j)]$$
$$- (1-k)j[\Delta_{m+1-j}(b_j) - \Delta_{m-j}(b_j)] = 0,\ j = 1, ..., d. \quad (6)$$

As in the discriminatory auction example, an analytic solution to this system is unattainable. In order to understand differential shading in this environment, let us return to the example of the last section, examined now in terms of the special case of $k = 1$ where the first order conditions are

$$[v(j) - b_j]\delta_{m+1-j}(b_j) - (j-1)[\Delta_{m+2-j}(b_j) - \Delta_{m+1-j}(b_j)] = 0,\ \ j = 1, ..., d. \quad (7)$$

When bidders only demand one unit each ($j = 1$), Eq. (7) gives the dominant strategy of placing a high bid equal to ones high valuation. However, Eq. (7) also shows that for $j > 1$, bids should be shaded below valuations. As opposed to bids in the discriminatory auction, differential shading in this case consists of low bids that are shaded more than high bids: a result deemed *demand reduction* by Ausubel and Cramton (2002).

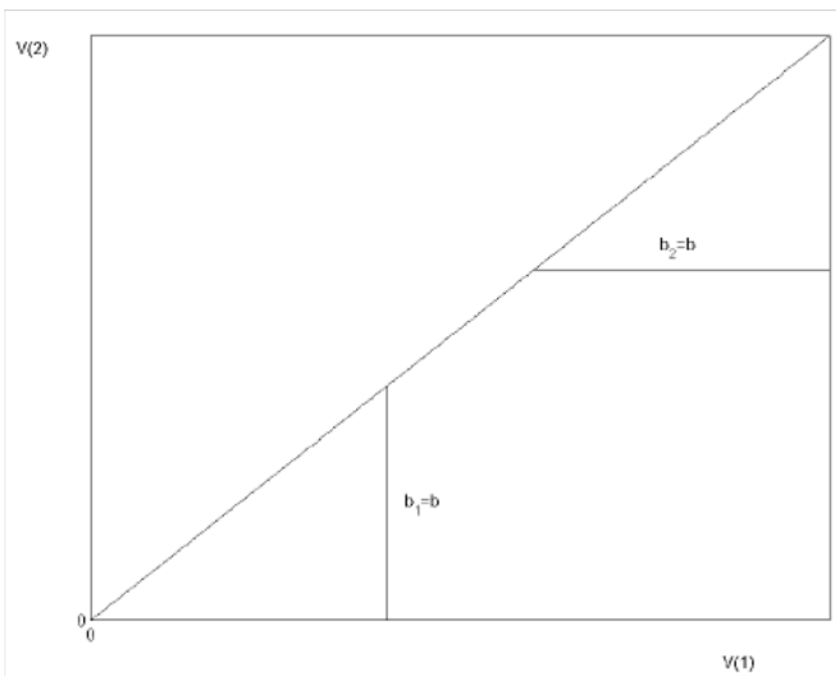

Figure 2: Uniform Price Auction Level Curves

Differential shading can be seen in Figure 2 as the high and low bid level curves do not meet at the 45^o line.

5. Asymptotic Results

The methodology used in this section is quite simple. Using the first order conditions derived in Sections 3 and 4, the level of bid shading is isolated and examined as $N \to \infty$. Price taking will emerge if all bids approach their corresponding valuations. Before proceeding we note that while the model was set up for any increasing bid functions used by a bidder's opponents, in this section we simply assume that opponents are bidding their valuations (which is a strictly increasing function itself) and ask if, in the limit, it is optimal for the optimizer to do so as well.

Lemma 1 *Given the assumptions on $F(v)$ and opponent bid functions, the expression $\frac{\Delta_{m+1-j}(b_j)}{\delta_{m+1-j}(b_j)}$ approaches zero as $N \to \infty$.*

Proof. Eq. (1) gives

$$\Delta_{m+1-j}(b_j) - \Delta_{m-j}(b_j) = \sum_{\sum i\lambda_i = m-j} \left\{ \binom{N}{\lambda_1, ..., \lambda_d} \prod_{w=0}^{d} \alpha_w(b)^{\lambda_w} \right\} \tag{8}$$

and Eq. (2) gives

$$\delta_{m+1-j}(b_j) = \delta_{m-j}(b_j) + \sum_{l=0}^{d} \sum_{\sum i\lambda_i = m-j} \left\{ \binom{N}{\lambda_1, ..., \lambda_d} \frac{\alpha_l'(b_j)\lambda_l}{\alpha_l(b_j)} \prod_{w=0}^{d} \alpha_w(b)^{\lambda_w} \right\}, \tag{9}$$

where $\lambda_0 = N - \sum_{j=1}^{d} \lambda_j$.

The laws of limits allow us to make a number of simplifications by making a few useful observations. (1) any summation using the combinatorial term $\binom{N}{\lambda_1,\ldots,\lambda_{m-j}}$, the term with the coefficient $\binom{N}{m-j,0,\ldots,0} = \binom{N}{m-j}$ will contain $N^{m-j}/(m-j)!$, the highest order term in N possible. (2) $\alpha_0(b)$ is the only α term raised to a power dependent on N. (3) dividing the numerator and denominator of $\frac{\Delta_{m+1-j}(b_j)}{\delta_{m+1-j}(b_j)}$ by $N^{m-j}\alpha_0(b)^{N-m+j}$ has two affects when taking the limit as $N \to \infty$. First, dividing by $\alpha_0(b)^{N-m+j}$ removes all dependence on N from the α probabilities and therefore only the combinatorial terms will be affected as $N \to \infty$. Second, because every N in the combinatorial terms is raised to a power less than or equal to $m-j$, dividing by N^{m-j} and taking the limit as $N \to \infty$ will cause all terms of order lower than $m-j$ to vanish. Dividing the numerator and denominator by $N^{m-j}\alpha_0(b)^{N-m+j}$ and taking the limit as $N \to \infty$, the numerator becomes $\alpha_1(b)^{m-j}$, a finite value. Likewise, the denominator becomes

$$\lim_{N\to\infty} \left\{ (N-m+j)\, \frac{\alpha_1(b)^{m-j}}{\alpha_0(b)} \alpha_0'(b) + (m-j)\, \alpha_1(b)^{m-j-1} \alpha_1'(b) \right\},$$

the limit of which clearly goes to infinity since $\alpha'(b) \neq 0$. ■

Theorem 2 *Differential shading vanishes in the discriminatory auction as $N \to \infty$ and price taking results.*

Proof. Rearranging Eq. (4) gives

$$(v(j) - b_j) = \frac{\Delta_{m+1-j}(b_j)}{\delta_{m+1-j}(b_j)},\ j = 1, \ldots, d. \tag{10}$$

The LHS of Eq. (10) can be seen as the amount that bid j is shaded below valuation $v(j)$. Lemma 1 shows that the RHS of Eq. (10) approaches zero as $N \to \infty$ and therefore, all bids approach their corresponding valuations. Since all bids eventually equal valuations, differential shading vanishes and the constraints do not bind in the limit. ■

Remark 1. The convergence of bids to valuations in the discriminatory auction is easily seen in Figure 1. As the number of bidders increases, the shaded region shrinks until the point at which the high and low bid level curves meet approaches the 45^o line.

Lemma 3 *Given the assumptions on $F(v)$ and opponent bid functions, the expression $j\frac{[\Delta_{m+1-j}(b_j)-\Delta_{m-j}(b_j)]}{\delta_{m+1-j}(b_j)}$ approaches zero as $N \to \infty$.*

Proof. A few useful observations are in order. (1) the term $[\Delta_{m+1-j}(b_j) - \Delta_{m-j}(b_j)]$ reduces to:

$$\sum_{\sum i\lambda_i = m-j} \binom{N}{\lambda_1, \ldots, \lambda_d} \prod_{w=0}^{d} \alpha_w(b)^{\lambda_w}. \tag{11}$$

(2) the term with the coefficient $\binom{N}{m-j}$ will be of the highest order in N. (3) Dividing the numerator and denominator of $\frac{[\Delta_{m+2-j}(b_j)-\Delta_{m+1-j}(b_j)]}{\delta_{m+1-j}(b_j)}$ by $N^{m-j}\alpha_0(b)^{N-m+j}$ has two effects. First, since all $\alpha_0(b)$ terms are raised to a power dependent on N, dividing by $\alpha_0(b)^{N-m+j}$ will remove all dependence on N from the α probabilities and the taking the limit will only affect the combinatorial terms. Second, dividing by N^{m-j} will cause all terms in which N is raised to a power less than $m-j$ to vanish in the limit. Dividing the numerator and denominator by $N^{m-j}\alpha_0(b)^{N-m+j}$ and taking the limit as $N \to \infty$, the numerator becomes $\alpha_1(b)^{m-j}$, a finite value. Likewise, the denominator becomes

$$\lim_{N\to\infty}\left\{(N-m+j)\,\frac{\alpha_1(b)^{m-j}}{\alpha_0(b)}\alpha_0'(b)+(m-j)\,\alpha_1(b)^{m-j-1}\alpha_1'(b)\right\},$$

the limit of which clearly goes to infinity since $\alpha'(b)\neq 0$. ■

Lemma 4 *Given the assumptions on $F(v)$ and opponent bid functions, the expression $(j-1)\frac{[\Delta_{m+2-j}(b_j)-\Delta_{m+1-j}(b_j)]}{\delta_{m+1-j}(b_j)}$ approaches $\frac{(j-1)\alpha_1(b)}{(m+1-j)\alpha_0'(b)}$ as $N \to \infty$.*

Proof. A few useful observations are in order. (1) the term $[\Delta_{m+2-j}(b_j) - \Delta_{m+1-j}(b_j)]$ reduces to:

$$\sum_{\sum i\lambda_i=m+1-j}\binom{N}{\lambda_1,\ldots,\lambda_d}\prod_{w=0}^{d}\alpha_w(b)^{\lambda_w}. \tag{12}$$

(2) the term with the coefficient $\binom{N}{m+1-j}$ will be of the highest order in N. (3) Dividing the numerator and denominator of $\frac{[\Delta_{m+1-j}(b_j)-\Delta_{m-j}(b_j)]}{\delta_{m+1-j}(b_j)}$ by $N^{m+1-j}\alpha_0(b)^{N-m-1+j}$ has two effects. First, since all $\alpha_0(b)$ terms are raised to a power dependent on N, dividing by $\alpha_0(b)^{N-m-1+j}$ removes all dependence on N from the α probabilities and taking the limit will only affect the combinatorial terms. Second, dividing by N^{m+1-j} will cause all terms in which N is raised to a power less than $m+1-j$ to vanish in the limit. Dividing the numerator and denominator by $N^{m+1-j}\alpha_0(b)^{N-m-1+j}$ and taking the limit as $N\to\infty$, reduces the numerator to $(j-1)\alpha_1(b)^{m+1-j}$, a finite number. Likewise, the denominator becomes $\alpha_1(b)^{m-j}\alpha_0'(b)$, a finite number

since $\alpha_0'(b) > 0$. The only way for differential shading to vanish in this case is for $\alpha'(b)$ to approach infinity. However, it cannot given the assumptions that opponents are bidding their valuations. ■

Theorem 5 *As $N \to \infty$, differential shading persists in the k-uniform price auction when $k \in (0, 1]$, but price taking emerges when $k = 0$.*

Proof. Rearranging Eq. (6) gives

$$\begin{aligned}(v(j) - b_j) &= (1-k)j\frac{[\Delta_{m+2-j}(b_j) - \Delta_{m+1-j}(b_j)]}{\delta_{m+1-j}(b_j)} \\ &\quad + k(j-1)\frac{[\Delta_{m+1-j}(b_j) - \Delta_{m-j}(b_j)]}{\delta_{m+1-j}(b_j)}. \end{aligned} \tag{13}$$

The LHS of Eq. (13) is the amount that bid j is shaded below valuation $v(j)$. Lemmas 3 and 4 show that only the first term on the RHS of Eq. (13) vanishes as $N \to \infty$ $\forall j$. Hence, as $N \to \infty$, the RHS will not vanish and bids will not equal values if $k \in (0, 1]$. ■

Remark 2. Not only does price taking fail to emerge, but since the non-zero convergent term is multiplied by a factor of $(j-1)$, bids are differentially shaded in the limit, indicating the possibility of ex post inefficiency. Swinkels (2001) has shown that the uniform price auction achieves an ex ante expectation of efficiency in the limit, leading to a seeming contradiction with our result. However, Swinkels' result allows for differential shading over certain regions of valuations, but hinges on the fact that it disappears over the "relevant" range of demand. The idea is that the probability that an inefficient bid wins approaches zero in the limit. We have shown that this ex ante expectation of an efficient allocation is not driven by price taking behavior.

6. Conclusions

The literature on single object auctions has provided a variety of results concerning revenue generation, allocative efficiency, and price formulation. The slow development of a cohesive theory of multi-unit auctions led economists to use these results to predict outcomes of their multi-unit counterparts (see Chari and Weber 1992). Unfortunately, recent theoretical advances have shown that the added complexities of a multi-unit environment lead to starkly different conclusions than those suggested by single object models. Despite these recent breakthroughs, the understanding of multi-unit auctions is far from complete. This paper enhances the understanding of multi-unit auctions by isolating the asymptotic properties of equilibria in IPV discriminatory and uniform price auctions. Our primary findings are that as the number of bidders grows large: (1) price taking behavior emerges in discriminatory auctions and uniform price auctions where the price paid equals the lowest winning bid and (2) price taking does not occur in any uniform price auction where the price is tied to the highest losing bid.

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