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Time-Dependent Perturbation and Exact Results for the Periodically Driven Quantum Harmonic Oscillator

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Time-dependent perturbation and exact results for the periodically driven quantum harmonic oscillator

Russell Akridge

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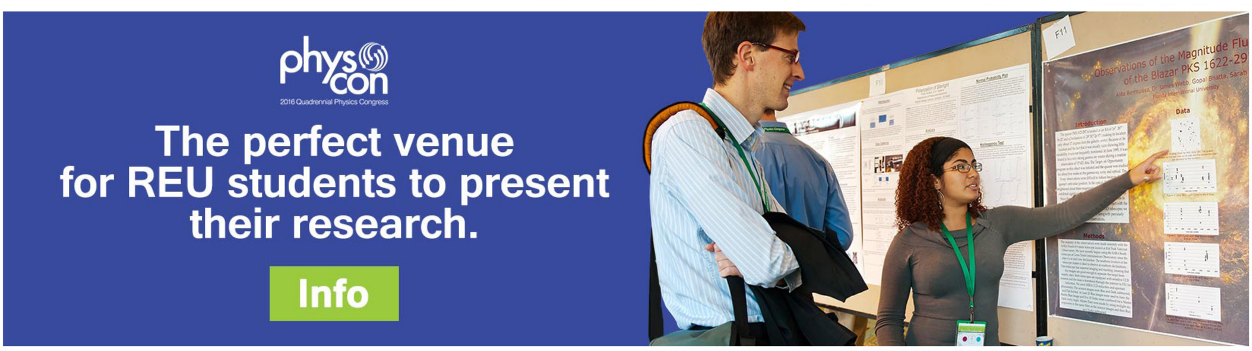

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resonance are large in amplitude, and the latter is long lived. The reflected packet comprises two components, the first mirrors the incident packet in shape and arises from the basic potential discontinuity, while the second has an exponential shape which arises directly from the decay of the intrawell wave-function resonance.

Our main interest here has been the shape of the reflected wave packet, and so we have not dwelt on features of this system such as the time decay of the intrawell resonance which is important for example in alpha particle emission. However we have found the movie technology very useful for demonstrating this and other related phenomena; such as the time delay on reflection from a potential step, and tunneling oscillation in a triple barrier/double well (analogous to the ammonia molecule).

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Time-dependent perturbation and exact results for the periodically driven quantum harmonic oscillator

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Transition amplitudes and probabilities for the harmonic oscillator with a forcing function proportional to $\cos(\omega t)$ beginning at time zero are calculated to lowest nonvanishing order using time-dependent perturbation theory. The results are compared with the exact amplitudes and probabilities. When the exact amplitude is expanded in a Taylor series in powers of the coupling constant, the individual terms turn out to be the perturbation amplitudes, showing that the complete series of perturbation amplitudes converges to the exact amplitude. © *1995 American Association of Physics Teachers.*

I. INTRODUCTION

Time-dependent perturbation theory is compared with exact results for the case of a periodically driven quantum oscillator.[1–8] The oscillator has mass m and unperturbed angular frequency ω_0, and it is in unperturbed eigenstate $\psi_n(x,t)$ when the perturbation is turned on. The periodic perturbing force is $mx_0\omega_0^2 \cos \omega t$, beginning at $t=0$. Time-dependent perturbation theory is used in Sec. II to calculate the lowest nonvanishing order transition amplitudes and probabilities. The actual form of the series of perturbation results can also be written down. Section III reviews the known exact solution. Section IV compares the transition amplitudes and probabilities calculated from perturbation theory with those calculated using the exact solutions. We find that a weak coupling does not necessarily guarantee good agreement between first nonvanishing order perturbation results and exact results. The approach to resonance plays a key factor in agreement or disagreement of the results. We also find out whether all transition probabilities sum to unity as they should in the exact and in the first nonvanishing order perturbation computation. Section V discusses transitions from the ground state to clarify the effect of resonance in both the final transition state and the time taken to reach that final state. Section VI presents the solution of all orders of perturbation transition amplitude between any two states. We find the perturbation series is an expansion of the exact result, as one would expect. Section VII summarizes the conclusions of the study.

II. TIME-DEPENDENT PERTURBATION RESULTS

The Hamiltonian for the harmonically driven oscillator is

$$H=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{2}m\omega_0^2x^2-m\omega_0^2x_0xf(t), \tag{2.1}$$

where m is the oscillator mass, ω_0 is the unperturbed oscillator angular frequency, and the time dependence of the driving force is

$$f(t)=\cos \omega t \tag{2.2}$$

when $t\geqslant 0$ and $f(t)=0$ for $t<0$. The well-known orthonormal solutions to the unperturbed Hamiltonian

$$H_0=-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\frac{1}{2}m\omega_0^2x^2 \tag{2.3}$$

are[9]

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$$\psi_n(x,t)=N_nH_n(\alpha x)\exp\left(-\frac{1}{2}\alpha^2x^2-i\frac{E_n}{\hbar}t\right), \quad (2.4a)$$

$$E_n=(n+\tfrac{1}{2})\hbar\omega_0, \quad (2.4b)$$

where

$$\alpha=\sqrt{\frac{m\omega_0}{\hbar}} \quad (2.5)$$

and

$$N_n=\sqrt{\frac{\alpha}{\pi^{1/2}2^n n!}} \quad (2.6)$$

and $H_n(\xi)$ is the Hermite polynomial of order n.

Then the perturbation Hamiltonian, H', for Eq. (2.1) is

$$H'=-m\omega_0^2x_0xf(t) \quad (2.7)$$

which can be written as the product of a term $\hbar\omega_0$ having the units of energy times a product of dimensionless factors

$$H'=-(\hbar\omega_0)(\alpha x)(\alpha x_0)f(t),$$

with α given by Eq. (2.5). Using

$$\lambda=\alpha x_0 \quad (2.8)$$

as a dimensionless form of the coupling strength, we can write

$$H'=-\lambda(\hbar\omega_0)(\alpha x)f(t). \quad (2.9)$$

If the oscillator is in the unperturbed eigenstate $\psi_n(x)e^{-i(E_n/\hbar)t}$ at time $t=0$, the amplitude for a transition to state j at time t is defined by the expansion $\Psi_n(x,t)=\Sigma_{j=0}^{\infty}a_{n,j}\psi_j(x,t)$, and is

$$a_{n,j}(t)=\int_{-\infty}^{+\infty}\psi_j^*(x,t)\Psi_n(x,t)dx, \quad (2.10)$$

where $\Psi_n(x,t)$ is the exact solution to $H\Psi=i\hbar(\partial\Psi/\partial t)$, with H given by Eq. (2.1), that reduces to Eq. (2.4a) at time $t=0$. When the perturbing term H' is small, because the interaction strength λ is small ($\lambda\ll1$), the exact transition amplitude can be expanded in a series in λ

$$a_{n,j}(t)=\sum_{s=0}^{\infty}\lambda^s a_{n,j}^{(s)}(t). \quad (2.11)$$

The standard formulas of perturbation theory give successive orders of perturbation by use of the formula[10]

$$\dot{a}_{n,j}^{(s+1)}=\frac{1}{i\hbar}\sum_r a_{n,r}^{(s)}(t)\frac{H'_{j,r}}{\lambda}\exp\left(i\frac{(E_r-E_j)}{\hbar}t\right) \quad (2.12)$$

and the matrix element term $H'_{j,r}/\lambda$ is independent of λ. For the unperturbed oscillator eigenfunctions the only nonzero matrix elements are[11]

$$\frac{H'_{j,j-1}}{\lambda}=-\sqrt{\frac{j}{2}}\hbar\omega_0\cos\omega t,$$

$$\frac{H'_{j,j+1}}{\lambda}=-\sqrt{\frac{j+1}{2}}\hbar\omega_0\cos\omega t$$

so the perturbation iterative Eq. (2.12) becomes

$$\dot{a}_{n,j}^{(s+1)}=i\omega_0\cos\omega t\left(\sqrt{\frac{j}{2}}a_{n,j-1}^{(s)}e^{i\omega_0t}+\sqrt{\frac{j+1}{2}}a_{n,j+1}^{(s)}e^{-i\omega_0t}\right). \quad (2.13)$$

It is relatively simple to find the lowest order nonvanishing amplitude from Eq. (2.13). We ensure that the oscillator is in the state n at time $t=0$ by choosing

$$a_{n,n}^{(0)}=1,\quad a_{n,j}^{(0)}=0\quad\text{for } j\neq n. \quad (2.14)$$

Thus, to zeroth order, the oscillator is in state n.

Consider a transition to a state $j=n+k$ higher than the original state n. According to Eq. (2.13) each successively higher order of perturbation s populates one higher quantum state. Therefore, the state $j=n+k$ will first be populated by a perturbation amplitude of order $s=k$. Further, only states that go up from the original state n succeed in populating state $j=n+k$ first, and we need retain only the first term of Eq. (2.13). For our purpose then Eq. (2.13) is

$$\dot{a}_{n,n+s}^{(s)}=\sqrt{\frac{n+s}{2}}(i\omega_0\cos\omega te^{i\omega_0t})a_{n,n+s-1}^{(s-1)}. \quad (2.15)$$

This formula can be simplified further using

$$\xi(t)=\beta(t)+i\frac{\dot{\beta}(t)}{\omega_0}, \quad (2.16)$$

where

$$\beta(t)=\frac{x_c(t)}{x_0}, \quad (2.17)$$

and $x_c(t)$ is the solution of the classical driven oscillator which is at rest at the origin until time zero, so that

$$\ddot{\beta}+\omega_0^2\beta-\omega_0^2f(t)=0. \quad (2.18)$$

This Newtonian equation of motion is derived from a dimensionless form of the classical Lagrangian $L(t)$ which is an important part of the exact solution discussed in Sec. III to follow

$$L(t)=\frac{\dot{\beta}^2}{2\omega_0^2}-\frac{\beta^2}{2}+\beta f(t). \quad (2.19)$$

It is straightforward to show that

$$\frac{d}{dt}(\xi(t)e^{i\omega_0t})=i\omega_0e^{i\omega_0t}\cos\omega t \quad (2.20)$$

by direct substitution from Eqs. (2.16) and (2.18). Now the equation for the lowest order nonvanishing transition amplitude, Eq. (2.15), becomes

$$\frac{d}{dt}(a_{n,n+s}^{(s)})=\sqrt{\frac{n+s}{2}}a_{n,n+(s-1)}^{(s-1)}\frac{d}{dt}[\xi(t)e^{i\omega_0t}]. \quad (2.21)$$

Beginning with $a_{0,0}^{(0)}=1$, this formula is easily integrated and iterated to yield[12]

$$a_{n,n+k}^{(k)}=\sqrt{\frac{(n+k)!}{2^kn!}}\frac{(\xi e^{i\omega_0t})^k}{k!},\quad k\geq0. \quad (2.22a)$$

Similarly the lowest order nonvanishing transition amplitude for states $j=n-k$ ($k<n+1$) below the original quantum state n is found to be

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$$a^{(k)}_{n,n-k}=(-1)^k\sqrt{\frac{n!}{2^k(n-k)!}}\,\frac{(\xi^* e^{i\omega_0 t})^k}{k!},\quad 0\leqslant k\leqslant n. \qquad (2.22b)$$

These amplitudes give the lowest order nonvanishing transition probabilities

$$P^{(k)}_{n,n+k}=\lambda^{2k}\frac{(n+k)!}{2^k n!(k!)^2}\left(\beta^2+\frac{\dot{\beta}^2}{\omega_0^2}\right)^k,\quad k\geqslant 0, \qquad (2.23a)$$

$$P^{(k)}_{n,n-k}=\lambda^{2k}\frac{n!}{2^k(n-k)!(k!)^2}\left(\beta^2+\frac{\dot{\beta}^2}{\omega_0^2}\right)^k,\quad 0\leqslant k\leqslant n. \qquad (2.23b)$$

We can show that the exact series (2.11) does not contain all powers of λ but only powers λ^{k+2r}, $r=0,1,2,\ldots$. Equations (2.13) and (2.14) yield $a^{(1)}_{0,0}=0$, $a^{(2)}_{0,0}\neq 0$. This zero for $a^{(1)}_{0,0}$ works its way up or down the quantum states given by Eq. (2.13) so that only every other state after the lowest order nonvanishing state is nonzero. Thus, we decide that the actual series in Eq. (2.11) is of the form

$$a_{n,n\pm k}=\sum_r \lambda^{k+2r}a^{(k+2r)}_{n,n\pm k} \qquad (2.24)$$

with the leading term $a^{(k)}_{n,n\pm k}$ given by Eq. (2.22a) or Eq. (2.22b).

III. EXACT SOLUTION

Nagomi[13] recently listed in this journal his exact solution to Eq. (2.1) which reduces to $\psi_n(x)\exp^{-iE_n/\hbar}$ as $t\to 0^+$ as

$$\Psi_n(x,t)=\psi_n[x-x_0\beta(t)]\exp\left\{i\left[-n\omega_0 t+\lambda^2\left(\frac{\dot{\beta}(x/x_0-\beta)}{\omega_0}+\int_0^t L(t')dt'\right)\right]\right\} \qquad (3.1)$$

and this solution has been known in one form or another since the 1950s.[14] Using this exact solution in Eq. (2.10) gives the exact transition amplitudes:

$$a_{n,j}=(\pi 2^{n+j}n!j!)^{-1/2}\exp\left\{-\frac{\lambda^2}{4}\left|\xi\right|^2\right\}\times\exp\left[i\left((j-n)\omega_0 t-\frac{\lambda^2\beta\dot{\beta}}{2\omega_0}+\int_0^t L(t')dt'\right)\right]\times\int_{-\infty}^{+\infty}H_n(y-\lambda\beta)H_j(y)\exp^{-(y-\lambda\xi/2)^2}dy \qquad (3.2)$$

using $y=\alpha x$. The integral is evaluated using the tabulated integral formula

$$\int_{-\infty}^{+\infty}e^{-\rho^2}H_n(\rho+\alpha)H_j(\rho+\beta)d\rho=2^j\pi^{1/2}n!\beta^{j-n}L_n^{j-n}(-2\alpha\beta),\quad j\geqslant n \qquad (3.3)$$

and a change of variable $\rho=y-\lambda\xi/2$. In this formula $L_n^m(u)$ is the associated Laguerre polynomial. Simplifying gives[15]

$$a_{n,n+k}=\lambda^k\sqrt{\frac{n!}{2^k(n+k)!}}\,\xi^k L_n^k\left(\frac{\lambda^2}{2}|\xi|^2\right)\exp\left\{i\left[k\omega_0 t+\lambda^2\left(\int_0^t L(t')dt'-\frac{\beta\dot{\beta}}{2\omega_0}\right)\right]-\frac{\lambda^2}{2}|\xi|^2\right\},\quad 0\leqslant n\leqslant k. \qquad (3.4a)$$

The analogous result for states below state n is

$$a_{n,n-k}=\lambda^k(-1)^k\sqrt{\frac{(n-k)!}{2^k n!}}\,(\xi^*)^k L^k_{n-k}\left(\frac{\lambda^2}{2}|\xi|^2\right)\times\exp\left\{i\left[k\omega_0 t+\lambda^2\left(\int_0^t L(t')dt'-\frac{\beta\dot{\beta}}{2\omega_0}\right)\right]-\frac{\lambda^2}{2}|\xi|^2\right\},\quad 0\leqslant k\leqslant n. \qquad (3.4b)$$

The exact transition probabilities are

$$P_{n,n+k}=\frac{n!}{(n+k)!}\left(\frac{\lambda^2}{2}|\xi|^2\right)^k\left[L_n^k\left(\frac{\lambda^2}{2}|\xi|^2\right)\right]^2\times\exp-\frac{\lambda^2}{2}|\xi|^2,\quad 0\leqslant n\leqslant k \qquad (3.5a)$$

or

$$P_{n,n-k}=\frac{(n-k)!}{n!}\left(\frac{\lambda^2}{2}|\xi|^2\right)^k\left[L^k_{n-k}\left(\frac{\lambda^2}{2}|\xi|^2\right)\right]^2\times\exp-\frac{\lambda^2}{2}|\xi|^2,\quad 0\leqslant k\leqslant n. \qquad (3.5b)$$

IV. COMPARISONS

Using perturbation theory, we discovered that the lowest nonvanishing order of $a^{(s)}_{n,n\pm k}$ was order $s=k$. This discovery is born out by the exact formulas (3.4) which are proportional to λ^k.

Using perturbation theory, we calculated the lowest order nonvanishing amplitudes and probabilities in Eqs. (2.22) and (2.23). We compare with the exact results when λ is set equal to zero in the imaginary exponential and in the associated Laguerre polynomial. This corresponds to the first term in an expansion of the exact result in a power series in λ. In this process, the leading λ^k in Eqs. (3.4) and λ^{2k} in Eqs. (3.5) is not set equal to zero, because these λ's are not part of the expansion in powers of λ. This lowest order term in the exact result is

$$[a_{n,n+k}]_0=\lambda^k e^{ik\omega_0 t}\sqrt{\frac{n!}{2^k(n+k)!}}\,\xi^k L_n^k(0),\quad 0\leqslant n\leqslant k \qquad (4.1a)$$

or

$$[a_{n,n-k}]_0=\lambda^k(-1)^k e^{-ik\omega_0 t}\sqrt{\frac{(n-k)!}{2^k n!}}\,(\xi^*)^k L^k_{n-k}(0),\quad 0\leqslant k\leqslant n. \qquad (4.1b)$$

Use of the tabulated associated Laguerre polynomial with zero argument[16]

$$L_n^a(0)=\frac{(n+a)!}{n!a!} \qquad (4.2)$$

gives the $[a]_0$ in the form

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$$[a_{n,n+k}]_0=\lambda^k \frac{e^{ik\omega_0 t}}{k!} \xi^k \sqrt{\frac{(n+k)!}{2^k n!}}, \qquad (4.3a)$$

$$[a_{n,n-k}]_0=\lambda^k(-1)^k \frac{e^{-ik\omega_0 t}}{k!} (\xi^*)^k \sqrt{\frac{n!}{2^k(n-k)!}}. \qquad (4.3b)$$

Comparing with Eqs. (2.22) we see that

$$[a_{n,n\pm k}]_0=\lambda^k a_{n,n\pm k}^{(k)}. \qquad (4.4)$$

The lowest nonvanishing result calculated from perturbation theory is identical to the first term in the expansion of the exact result in a power series in the coupling parameter.

We determined from perturbation theory that the perturbation series would be of the form of a leading term in λ^k times a series in only even powers of λ as in Eq. (2.24). Comparing with the exact result in Eq. (3.4) we see that when the exact result is expanded in a series, it will contain a leading term of λ^k times a series in integer powers of λ^2, confirming the perturbation result.

The contrast between the exact and the first nonvanishing order transition probabilities is easily seen by examining transitions from the ground state $n=0$ to higher states. We have from Eqs. (2.23a) and (3.5a)

$$P_{0,k}^{(k)}=\frac{u^k}{k!}, \qquad (4.5a)$$

$$P_{0,k}=\frac{u^k}{k!} e^{-u}, \qquad (4.5b)$$

where

$$u=\frac{\lambda^2|\xi|^2}{2}. \qquad (4.6)$$

In the spirit of perturbation theory, $\lambda \ll 1$, so $0<u \ll 1$. Unless $|\xi|^2$ is huge, transitions from the ground state are unlikely. The classical oscillator solution to Eq. (2.18) for the stated initial conditions is

$$\beta(t)=\frac{(\cos \omega t-\cos \omega_0 t)}{\left(1-\frac{\omega^2}{\omega_0^2}\right)},$$

so u varies as

$$u \propto |\xi|^2 \propto \frac{1}{(1-\omega^2/\omega_0^2)^2}; \qquad (4.7)$$

$|\xi|^2$ is huge only near resonance, $\omega \approx \omega_0$, as would be expected classically.

Away from resonance, the exact and the first nonvanishing order probability results in Eqs. (4.5) are nearly the same, because $e^{-u} \approx 1$ for $|u| \ll 1$. However, for $k \neq 0$, both formulas show that all excited states are sparsely populated. The oscillator is most likely to remain in the ground state. The perturbation result does not sum to unity properly, since $P_{0,0}^{(0)}=1$ and $P_{0,k}^{(0)}>0$ for all $k>0$. Since each $|P_{0,k}^{(0)}|$ is small, the sum might be expected to be close to 1. In fact the sum is easy to carry out

$$\sum_{k=0}^{\infty} P_{0,k}^{(0)}=\sum_{k=0}^{\infty} \frac{u^k}{k!}=e^u \qquad (4.8)$$

which is approximately 1, since $|u| \ll 1$ far from resonance. The exact amplitudes do sum to 1 as they should

$$\sum_{k=0}^{\infty} P_{0,k}=\sum_{k=0}^{\infty} \frac{u^k}{k!} e^{-u}=e^u e^{-u}=1. \qquad (4.9)$$

Near resonance, the first nonvanishing order perturbation transition probability departs from the exact transition probability. The departure is more severe the closer resonance is approached. The departure occurs, even though the coupling constant λ is small, $|\lambda| \ll 1$, because $|\lambda u/2|>1$ near resonance. Equations (4.5a) and (4.5b) show that both the exact and the first nonvanishing order transition probabilities now predict significant population among the excited states. The perturbation result overestimates the transition probability by the factor of e^u which now is a large factor. The ratio of transition probabilities for different states is the same for both results. The exact transition probabilities sum to 1 as they should, but the first-order nonvanishing transition probabilities sum to e^u which is now much larger than 1.

In general, both the exact and the first nonvanishing order transition probabilities compare favorably as long as $|u| \ll 1$. However, the perturbation result departs significantly from the exact result when $|u| \gg 1$ as resonance is approached, even though the coupling constant $\lambda=\alpha x_0$ is tiny ($|\lambda| \ll 1$).

In light of these very different behaviors, the next section is devoted to an analysis of the transition probability from the ground state as it depends on the magnitude of u.

V. EXCITATIONS FROM THE GROUND STATE

The transition probability from the ground state to any higher state n given by Eq. (4.5b) has the form of a Poisson distribution:[17]

$$P_n(u)=\frac{u^n}{n!} e^{-u}, \quad u=\frac{(\alpha x_0)}{2}\left(\beta^2+\frac{\dot{\beta}^2}{\omega_0^2}\right). \qquad (5.1)$$

This distribution has well-known limits. If $u<1$ (and $u>0$ always), then $P_0(u)$ is the largest of all of the $P_n(u)$ and $P_{n+1}(u)<P_n(u)$ for all $n \geq 0$. The ground state is the only state with significant population. The higher the energy of the state, the less likely it is to be excited.

If $u \gg 1$, the Poisson distribution is approximated by the standard normal distribution[18]

$$P_n(u) \approx \frac{\exp[-(n-u)^2/2u]}{\sqrt{2\pi u}}. \qquad (5.2)$$

The state most likely populated is the state $\bar{n}$ for which

$$\bar{n}=u, \qquad (5.3)$$

and the standard normal distribution of Eq. (5.2) is distributed around this state.

Which of these two limits applies depends on the value of u in Eq. (5.1). In general u will be a rapidly fluctuating function of time, so we use its time average value from Eq. (5.1) and the solution

$$\beta(t)=\frac{(\cos \omega t-\cos \omega_0 t)}{(1-\omega^2/\omega_0^2)}$$

of Eq. (2.18) and its initial condition. In the averaging process we replace the average value of $\cos^2$ and $\sin^2$ by 1/2, and we replace the average value of products of trig functions of different arguments by zero

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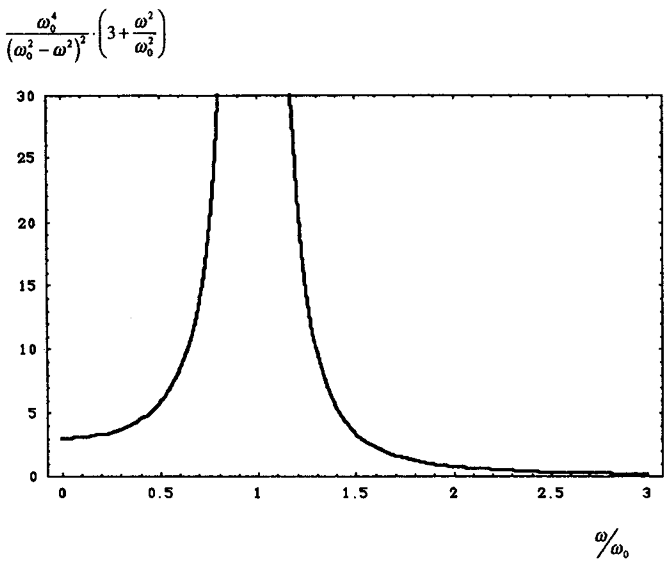

Fig. 1. The frequency dependence of the average state most likely to be populated, from Eq. (5.4).

$$u_{\text{ave}}=\left(\frac{(\alpha x_0)^2}{4}\right)\left[\frac{\omega_0^4}{(\omega_0^2-\omega^2)^2}\left(3+\frac{\omega^2}{\omega_0^2}\right)\right]. \tag{5.4}$$

Note that we have written this average value as a dimensionless interaction strength term times a dimensionless frequency term to separate the effects of the interaction strength and the frequency. The frequency dependence is plotted in Fig. 1.

The interaction strength term $(\alpha x_0)^2/4$ could have any numerical value. However, the term in x_0 in the original Schrödinger Eq. (2.1) was considered a perturbation. Since $|\alpha x|\approx 1$ characterizes the spatial extent of the unperturbed oscillator, we consider only the case in which $\alpha x_0<1$. This way our interaction perturbation is weak in the spirit of perturbation theory. This requirement of a weak perturbation produces a leading term in Eq. (5.4) that is much less than unity. If u_{ave} is ever to be large enough to populate higher states, the frequency dependent term must be very large. Its limits for large and small ω are

$$\frac{\omega_0^4}{(\omega_0^2-\omega^2)^2}\left(3+\frac{\omega^2}{\omega_0^2}\right)\xrightarrow[\omega\gg\omega_0]{}\left(\frac{\omega_0}{\omega}\right)^2\to 0 \tag{5.5a}$$

and

$$\frac{\omega_0^4}{(\omega_0^2-\omega^2)^2}\left(3+\frac{\omega^2}{\omega_0^2}\right)\xrightarrow[\omega\ll\omega_0]{} 3. \tag{5.5b}$$

The only hope of a large enough u_{ave} to populate excited states is the only remaining possibility, $\omega\approx\omega_0$, the resonance condition. Let

$$\omega=\omega_0+\Delta\omega, \quad \left|\frac{\Delta\omega}{\omega_0}\right|\ll 1. \tag{5.6}$$

Then

$$\frac{\omega_0^4}{(\omega_0^2-\omega^2)^2}\left(3+\frac{\omega^2}{\omega_0^2}\right)\xrightarrow[|\Delta\omega/\omega_0|\ll 1]{}\left(\frac{\omega_0}{\Delta\omega}\right)^2. \tag{5.7}$$

When the perturbing frequency nearly matches the natural oscillator frequency, u_{ave} can become so huge that higher states are populated even though the coupling $(\alpha x_0)^2/4$ is weak

$$u_{\text{ave}}\xrightarrow[|\Delta\omega/\omega_0|\ll 1]{}\frac{1}{4}(\alpha x_0)^2\left(\frac{\omega_0}{\Delta\omega}\right)^2. \tag{5.8}$$

The most likely quantum number $\bar{n}$ to be excited by this Poisson distribution is

$$\bar{n}=u_{\text{ave}}=\frac{1}{4}(\alpha x_0)^2\left(\frac{\omega_0}{\Delta\omega}\right)^2, \tag{5.9}$$

and the most likely energy[19] to be excited is

$$\bar{E}=\frac{1}{4}\hbar\omega_0(\alpha x_0)^2\left(\frac{\omega_0}{\Delta\omega}\right)^2. \tag{5.10}$$

Using the full width at half-maximum as the measure of the width of the excited states we have

$$\Delta\bar{E}=\hbar\omega_0(\alpha x_0)\left(\frac{\omega_0}{\Delta\omega}\right). \tag{5.11}$$

Higher states are excited significantly only if $\bar{n}\gg 1$, so Eq. (5.9) tells us how close the perturbing frequency must be to the natural frequency to excite state $\bar{n}$ as the most likely excited state

$$\Delta\omega=\frac{\omega\alpha x_0}{2\sqrt{\bar{n}}}. \tag{5.12}$$

In any case

$$\left|\frac{\Delta\omega}{\omega_0}\right|<\alpha x_0. \tag{5.13}$$

Weaker coupling αx_0 requires perturbing frequencies closer to the natural frequency in order to excite higher states, as could be expected physically.

A classical oscillator driven by a frequency very close to its natural frequency takes some time to absorb its energy, especially if the coupling is weak. We look for this effect in the driven quantum oscillator when $\omega\approx\omega_0$

$$\beta^2+\frac{\dot{\beta}^2}{\omega_0^2}\xrightarrow[\substack{|\Delta\omega/\omega_0|\ll 1\\ t\ll 1/\Delta\omega}]{}\frac{1}{4}[(\omega_0 t)^2+(\omega_0 t)\sin 2\omega_0 t+\sin^2\omega_0 t]. \tag{5.14}$$

The time dependence of the function on the left-hand side of Eq. (5.14) is shown in Figs. 2 and 3.

For all times $t\ll 1/\Delta\omega$, except the very smallest time $t<1/\omega_0$, Eq. (5.14) is effectively

$$\beta^2+\frac{\dot{\beta}^2}{\omega_0^2}\xrightarrow[\substack{|\Delta\omega/\omega_0|\ll 1\\ 1/\omega_0<t\ll 1/\Delta\omega}]{}\left(\frac{\omega_0 t}{2}\right)^2. \tag{5.15}$$

With t in the range $1/\omega_0<t\ll 1/\Delta\omega$, u from Eq. (5.1) becomes

$$u\xrightarrow[\substack{|\Delta\omega/\omega_0|\ll 1\\ 1/\omega_0<t\ll 1/\Delta\omega}]{}\frac{(\alpha x_0)^2(\omega_0 t)^2}{8}. \tag{5.16}$$

We use the Poisson condition $\bar{n}=u$ to find the excited states that are populated shortly after the driving interaction is turned on

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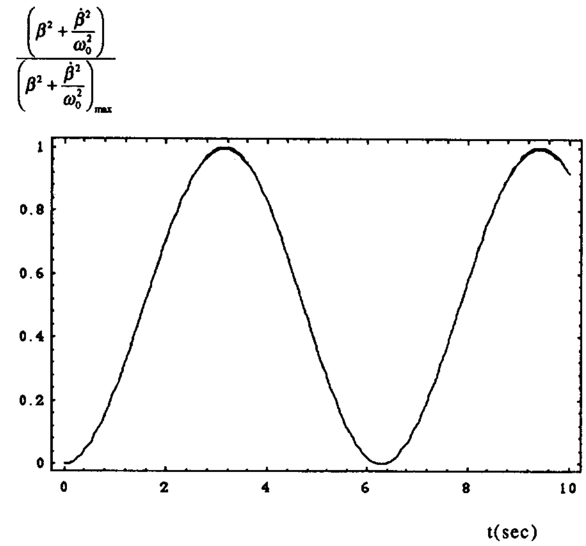

Fig. 2. Time dependence of the energy of the most likely state for a quantum driven oscillator near resonance from Eq. (5.18). For this graph $\omega_0=200$ rad/s, $\omega=201$ rad/s, and the graph is normalized to a maximum of unity. The period of the envelope is $2\pi/\Delta\omega=6.28$ s.

$$\bar{n}=\frac{(\alpha x_0)^2(\omega_0 t)^2}{8}, \quad \left(\frac{\Delta\omega}{\omega_0}\ll 1, \quad \frac{1}{\omega_0}<t\ll\frac{1}{\Delta\omega}\right). \quad (5.17)$$

See Fig. 3 to compare this approximate time dependence with the exact time dependence. After the interaction is turned on, only the ground state is significantly populated until time $t\approx\alpha x_0/\omega_0$. Then higher states begin to be populated. Unlike the classical oscillator, the quantum oscillator takes some time before it can absorb enough energy to make a quantum jump to higher energies. The most likely populated state moves higher and higher as t^2 as the perturbing interaction is left on in the resonance condition until $t\approx 1/\Delta\omega$, which could be a long time as resonance is approached.[20] After this time the oscillator energy is a periodic function of time.

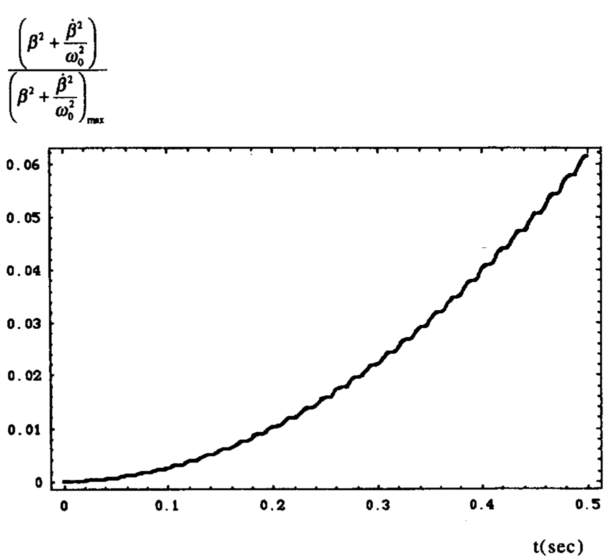

Fig. 3. The same as Fig. 2, except for small time. Compare the overall parabolic shape with the t^2 dependence of Eq. (5.16).

If we calculate the oscillator energy $E(t)=\bar{n}\hbar\omega_0$ and use $n=\bar{n}=u$ from Eq. (5.1) we get[21]

$$E(t)=\frac{(\hbar\omega_0)(\alpha x_0)}{2}\left(\beta^2+\frac{\dot\beta^2}{\omega_0^2}\right)=\frac{1}{2}mx_0^2\left(\beta^2+\frac{\dot\beta^2}{\omega_0^2}\right), \quad (5.18)$$

which is the same as the time dependent energy for the classical oscillator in accord with the correspondence principle.

VI. PERTURBATION AMPLITUDE TO ANY ORDER FROM ANY STATE TO ANY STATE

In Eq. (3.4a) the exponential is a series in λ^2 and the Laguerre polynomial is the series

$$L_n^k(\lambda^2\nu)=\sum_{m=0}^{n}(-1)^m\binom{n+k}{n-m}\frac{(\lambda^2\nu)^m}{m!}. \quad (6.1)$$

Multiplying this with the exponential series and collecting like terms, we have

$$e^{\lambda^2 u}L_n^k(\lambda^2\nu)=\sum_{r=0}^{\infty}\left[\frac{1}{r!}\sum_{s=0}^{s=\min(r,n)}(-1)^s\binom{n+k}{n-s}\times\binom{r}{s}u^{r-s}\nu^s\right]. \quad (6.2)$$

Using this identity in Eq. (3.4a) and comparing with Eq. (2.24) we have[22,23]

$$a_{n,n+k}^{(k+2r)}=\frac{(-1)^r}{r!}\sqrt{\frac{n!}{2^k(n+k)!}}\,\xi^k e^{ik\omega_0 t}\sum_{s=0}^{\min(r,n)}\binom{n+k}{n-s}\times\binom{r}{s}\left(\frac{|\xi|^2}{4}\right)^s\left[i\left(\int_0^t L(t')dt'-\frac{\beta\dot\beta}{2\omega_0}\right)-\frac{|\xi|^2}{2}\right]^{r-s}, \quad (6.3)$$

and this reduces to Eq. (2.22a) for $r=0$. In particular, for transitions from the ground state,

$$a_{0,k}^{(k+2r)}=\frac{(-1)^r}{r!}\frac{\xi^k e^{ik\omega_0 t}}{\sqrt{2^k k!}}\left[i\left(\int_0^t L(t')dt'-\frac{\beta\dot\beta}{2\omega_0}\right)-\frac{|\xi|^2}{2}\right]^r. \quad (6.4)$$

The result of a similar process for transitions to lower states is

$$a_{n,n-k}^{(k+2r)}=\frac{1}{r!}\sqrt{\frac{(n-k)!}{2^k n!}}\,(\xi^*)^k e^{ik\omega_0 t}\cdot\sum_{s=0}^{\min(r,n-k)}(-1)^{s+k}\binom{n}{n-k-s}\binom{r}{s}\times\left(\frac{|\xi|^2}{4}\right)^s\left[i\left(\int_0^t L(t')dt'-\frac{\beta\dot\beta}{2\omega_0}\right)-\frac{|\xi|^2}{2}\right]^{r-s} \quad (6.5)$$

and this reduces to Eq. (2.22b) for $r=0$, the lowest nonvanishing order.

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Each of these formulas is the complete solution to the perturbation Eq. (2.13) for all orders of perturbation from any state to any state, as can be verified by straightforward albeit laborious substitution into Eq. (2.13) and liberal use of identities.

VII. CONCLUSIONS

The lowest nonvanishing order perturbation transition amplitudes and probabilities for the harmonically driven oscillator are calculated for transitions between any two states. These lowest nonvanishing order results are the same as the leading term in an expansion of the exact transition amplitudes and probabilities. When the exact amplitudes are expanded in a series in powers of the perturbation coupling constant, the individual terms of the series satisfy the perturbation equation for that order. Thus, for any interaction strength, the perturbation series converges for this problem.

Lowest nonvanishing order perturbation results are close to exact results, except near resonance. Even for tiny interaction coupling, near resonance, exact and perturbation results are not close. For a weak interaction, few transitions occur, except near resonance, and then considerable time is required for those transitions to occur.

ACKNOWLEDGMENTS

I wish to acknowledge the assistance, suggestions, and helpful comments of Hal Gersch of Georgia Tech and Leon Combs of Kennesaw State College.

[1]G. W. Parker, "Evolution of the quantum states of a harmonic oscillator in a uniform time varying electric field," Am. J. Phys. **40**, 120–125 (1972); Parker develops a general formalism and solves the case of an exponentially varying interaction, $H' \propto xe^{-t/T}$.

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[10]Reference 9, Eq. (29.7).

[11]Reference 9, Eq. (13.18).

[12]Barry R. Holstein, *Topics in Advanced Quantum Mechanics* (Addison-Wesley, Redwood City, CA, 1992). Holstein lists the first and second order time-dependent perturbation results as a problem on p. 37. Note that Holstein's results are only for the cases of $k=1$ and $k=2$ in our Eqs. (2.22), whereas our equations (2.22) are valid for all possible values of the order of perturbation k.

[13]Y. Nagomi, "Test of the adiabatic approximation in quantum mechanics: Forced harmonic oscillator," Am. J. Phys. **59**, 64–8 (1991), especially Sec. III. $L(t')$ in this equation is the classical Lagrangian defined by Eq. (2.19), except that in Eq. (3.1) the Lagrangian is expressed in terms of t' instead of t.

[14]See the references 1–8 in Sec. I. Also Ref. 12, Holstein, pp. 236–42 derives a general form of the exact transition amplitudes which he acknowledges is due to Richard P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965). Holstein considers explicitly the exponential time dependence, but not the periodic time dependent driving force.

[15]These results are arrived at in a different way and presented in a different form in R. W. Fuller, S. M. Harris, and E. Leo Slaggie, "S-matrix solution for the forced harmonic oscillator," Am. J. Phys. **31**, pp. 431–439. In particular see their Eqs. (51) and (52).

[16]I. S. Gradshteyn, and I. M. Ryzhik, *Table of Integrals, Series, and Products*, trans. Alan Jeffrey (Academic, New York, 1965), p. 1038, Eq. 8.973.3.

[17]John E. Freud and Ronald E. Walpole, *Mathematical Statistics*, 4th ed. (Prentice-Hall, Englewood Cliffs, NJ, 1987), Sec. 5.7.

[18]Reference 17, Sec. 6.5 and 8.2.

[19]Here we assume that the quantum state n is very large so that we may neglect the $\frac{1}{2}\hbar\omega_0$ compared to $n\hbar\omega_0$ in the energy $E_n=(n+\frac{1}{2})\hbar\omega_0$.

[20]For the brief instant $0\leqslant t\leqslant 1/\omega_0$, Eq. (14) reduces to $\beta^2+\dot{\beta}^2/\omega_0^2\rightarrow(\omega_0 t)^2$, so $u\rightarrow\frac{1}{2}(\alpha x_0)^2(\omega_0 t)^2$. But the condition $\frac{1}{2}(\alpha x_0)^2(\omega_0 t)^2=u=\bar{n}$ for higher states to be populated cannot be satisfied, because then both $(\alpha x_0)\leqslant 1$, and $\omega_0 t<1$.

[21]Here $\bar{n}\gg 1$ so that $(\bar{n}+\frac{1}{2})\hbar\omega_0\approx\bar{n}\hbar\omega_0$.

[22]M. M. Ninan and Z. Stipcevic, "Transition probability of a linearly forced harmonic oscillator," Am. J. Phys. **37**, 734–6 (1969). These authors use a binomial theorem for noncommuting operators to derive $S_{j,k} = L_j^{k-j}(x)x^{k-j/2} \cdot \sqrt{j!/k!}\exp\{-x/2\}$ and $P_{j,k}=(j!/k!)x^{k-j}\times[L_j^{k-j}(x)]^2\exp(-x)$, the transition amplitude and probability respectively, where $x = (\int_{t'=0}^{t'=t}F(t')\exp(i\omega_0 t')dt')^2/[2(\alpha\hbar)^2]$. They do not calculate results for any particular driving force function $F(t)$.

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KNITTING ON THE JOB

When I operated Ray [Herb]'s machine for long runs there wasn't too much to do and I was bored. So I knitted—until the day Ray sent a notice to *all* members of the nuclear group stating that people who used the accelerator were, please, not to knit at the same time.

Fay Ajzenberg-Selove, *A Matter of Choices—Memoirs of a Female Physicist* (Rutgers University Press, New Brunswick, New Jersey, 1994), p. 75.

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