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ASPECTS OF CONFORMAL FIELD THEORY FROM CALABI-YAU ARITHMETIC[◇]

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ABSTRACT:

This paper describes a framework in which techniques from arithmetic algebraic geometry are used to formulate a direct and intrinsic link between the geometry of Calabi-Yau manifolds and aspects of the underlying conformal field theory. As an application the algebraic number field determined by the fusion rules of the conformal field theory is derived from the number theoretic structure of the cohomological Hasse-Weil L-function determined by Artin's congruent zeta function of the algebraic variety. In this context a natural number theoretic characterization arises for the quantum dimensions in this geometrically determined algebraic number field.

PACS Numbers and Keywords:

Math: 11G25 Varieties over finite fields; 11G40 L-functions; 14G10 Zeta functions; 14G40 Arithmetic Varieties

Phys: 11.25.-w Fundamental strings; 11.25.Hf Conformal Field Theory; 11.25.Mj Compactification

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1 Introduction

One of the results to emerge from the second period of string theory has been the discovery of mirror symmetry some ten years ago. The first large scale cohomological evidence of mirror symmetry presented in [9] was soon understood to be a consequence of a mirror construction that mapped Calabi-Yau orbifolds into complete intersection varieties [27]. Contemporaneously, a much more precise way of constructing mirror pairs by orbifolding exactly solvable string compactifications was discovered in ref. [19]. It is the conformal field theory picture which gives confidence to the conclusion that topologically rather different mirror pairs of Calabi-Yau varieties should define isomorphic compactifications of string theory. This insight has been used by physicists and mathematicians to derive a number of interesting consequences, such as the computation of the number of rational curves on Calabi-Yau varieties [5]. A review of many of the results can be found in [10].

At present, mirror symmetry, in all its geometric ramifications, is still poorly understood, and the conformal field theoretic constructions remain an important tool in its exploration. The simplest class of models in which conformal field theory has been linked to Calabi-Yau varieties is provided by Gepner's construction of tensor models of minimal $N = 2$ superconformal field theories [17]. When Gepner originally constructed non-geometric string compactifications with 2-dimensional interacting exactly solvable theories on the worldsheet his expectation was to have obtained an entirely new class of consistent string backgrounds. But when comparing the spectra of several of his models with the Hodge numbers of some existing Calabi-Yau manifolds [8, 32] he noticed that, surprisingly, there was agreement between the spectra of these rather different looking models [18]. Further work established a more systematic relation, initially based on a Landau-Ginzburg mean field theory description of the conformal field theory [29, 40, 25, 26, 16], and later on a sigma-theoretic analysis [43].

In the Landau-Ginzburg picture part of the structure of the conformal field theory is encoded in the chiral ring, determined by the superpotential, which for diagonal minimal models takes the form $W(\Phi_i) = \sum_{i=0}^{s} \Phi_i^{n_i}(z, \bar{z})$, where the $\Phi_i(z, \bar{z})$ are chiral primary fields on the string world sheet Riemann surface. In the sigma-theoretic picture both the Landau-Ginzburg theory and the Calabi-Yau variety are recovered as different phases of the model.

Both the Landau-Ginzburg and the sigma-theoretic approach illuminate the relation between Calabi-Yaus and conformal field theories. What is lacking, however, is a direct, intrinsic, and mathematically rigorous framework which allows to establish a link between algebraic varieties and conformal field theories. One might hope for a framework in which it is possible to derive the essential ingredients of the conformal field theory directly from the algebraic variety itself, without the intermediate Landau-Ginzburg formulation. A priori it might appear unlikely that such an approach exists because numbers associated to algebraic varieties usually are integers (such as dimension, cohomology dimensions and indices associated to complexes), whereas the numbers appearing in the underlying exactly solvable conformal field theory usually are rational numbers (such as the central charge and the anomalous dimensions).

This article describes a strategy to formulate an intrinsic, direct, and mathematically rigorous framework which allows to derive certain conformal field theoretic quantities directly from the algebraic Calabi-Yau variety [34]. The idea is to use concepts from arithmetic algebraic geometry, in which algebraic varieties X are defined not over a continuous field, like the real numbers $\mathbb{R}$, or the complex number field $\mathbb{C}$, but over discrete finite fields $\mathbb{F}_q$, where $q \in \mathbb{N}$ denotes the number of elements of the field. The particular field of reduction of the variety is specified by writing $X/\mathbb{F}_q$, leading to a reduced variety consisting of a finite number of points.

In the past it has not been particularly useful in physics to consider manifolds over fields other than the complex or real numbers. All work done in string compactification over the last fifteen years has been done over the complex numbers[1]. Lattice constructions usually are viewed as a preliminary step to taking some kind of continuum limit which is considered to describe the correct model. From a physical point of view the choice of any finite q in the reduction of a variety $X/\mathbb{F}_q$ would appear to be arbitrary and ill-motivated. For small q the field $\mathbb{F}_q$ would define a large scale lattice structure which one might expect to provide only rough information about the structure of the variety. More sensible is to consider an infinite sequence of ever

[1]A recent exception appears in these proceedings [6, 7].

larger finite fields which probe the variety at ever smaller scales. It is this consideration which leads us to the concept of counting the number of solutions $N_{r,p}(X) = \#(X/\mathbb{F}_{p^r})$ and to ask what, if any, interesting information is provided by the numbers determined by the extensions $\mathbb{F}_{p^r}$ of degree r of the finite field $\mathbb{F}_q$. The following paragraphs outline the strategy envisioned in this program and briefly introduce the arithmetic ingredients used in this paper.

The starting point of the arithmetic considerations in Section 4 is to arrange the sequence of reductions of the variety over the finite fields $\mathbb{F}_{p^r}$ into a useful form. This can be achieved via Artin's congruent zeta function, essentially defining an exponential sum of a generating function constructed from the numbers $N_{r,p}(X)$. For reasons just described we should not restrict the construction to local considerations at particular integers. Hence we need some way to pass to a global description in which all reference to fixed scales has been erased. This leads in Section 5 to the concept of the global Hasse-Weil L-function of a Calabi-Yau variety. This L-function will collect the information at all rational primes. This step can be taken because the Weil conjectures proven by Dwork and Deligne show that in the present context Artin's zeta function is a rational function determined by the cohomology of the variety. The Hasse-Weil L-function therefore is a cohomological L-function.

At this point the only number theoretic algebraic structures that have appeared are the finite fields $\mathbb{F}_p$ and their extensions. A brief review of some of the aspects of the underlying conformal field theoretic aspects, provided in Section 2, reveals that this is not enough. As mentioned above, the structures that enter in Gepner's construction and its generalizations are rational, i.e. they lead to central charges and spectra of anomalous dimensions which are rational numbers. It is not clear how to recover these numbers from the intrinsic geometry of the variety. It turns out that it is more useful in the present context to encode the conformal field theoretic information in an alternative way by mapping the central charge and the anomalous dimensions via the Rogers dilogarithm into the quantum dimensions associated to the physical fields. Section 3 reviews that these (generalized) quantum dimensions are elements of certain algebraic number fields which are determined by the fusion rules of the conformal field theory.

This suggests that we consider the structure of algebraic number fields in more detail. It is in this context that Hecke introduced a general notion of L-functions which are determined by the prime ideals of the number field, generalizing earlier results obtained by Dirichlet and Dedekind. Such number field L-functions, reviewed in Section 7, essentially count the inverse of the norms of ideals in the number field weighted by a character of this field.

The surprise, for the uninitiated, is that the cohomological Hasse-Weil type L-function of the Calabi-Yau variety in fact happens to be determined by number field L-functions of the type introduced by Hecke. This is the link that allows us to recover in Sections 7 and 8 the number theoretic framework relevant to the underlying conformal field theory directly from the intrinsic geometry of the variety. Once the fusion field has been identified one can then further explore the arithmetic rôle played by the quantum dimensions in this field.

The strategy used here can be summarized as follows. First consider the arithmetic structure of Calabi-Yau varieties via Artin's congruent zeta function. Next define the global cohomological Hasse-Weil L-function via local factors from the congruent zeta function. Finally interpret the Hasse-Weil L-series as Hecke L-series of an algebraic number field. In this way we can recover the algebraic fusion field of the conformal field theory from the algebraic variety.

The program to use arithmetic geometry to illuminate the conformal field theoretic structure of Calabi-Yau varieties originated several years ago. The idea initially was to develop further some results derived by Bloch and Schoen in support of the Beilinson-Bloch conjectures, and apply them in the context of the conformal field theory/Calabi-Yau relation [33]. The final section of the paper contains a brief outline of the concepts involved in framework. More recently arithmetic considerations have been discussed in different contexts in refs. [30] and [6].

2 Conformal Field Theory

2.1 Rational Theories

The simplest constructions of exactly solvable Calabi-Yau varieties involve conformal field theories with special features, described by rational exactly solvable field theories with N=2 supersymmetry.

These are defined by infinite dimensional Lie-algebras defined by the so-called Virasoro algebra defined in terms of the Fourier modes of the energy momentum tensor on the 2-dimensional theory

$$T(z) = \sum_{n \in \mathbb{Z}} \frac{L_n}{z^{n+2}}$$

for which the algebra takes the form

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}.$$

N=2 supersymmetry requires that the Virasoro algebra is extended by some affine Lie algebra currents

$$J(z) = \sum_{n \in \mathbb{Z}} \frac{J_n}{z^{n+1}},$$

e.g. $\mathrm{SU}(n)_k$ at level k, described by the commutator relation of its components $J_n = J_n^a T_a$

$$[J_m^a, J_n^b] = i f_c^{ab} J_{m+n}^c + km\delta_{m+n,0}\delta^{ab}.$$

Rationality of the theory means that the Hilbert space decomposes into a finite number of blocks

$$\mathcal{H} = \bigoplus_{\ell=1}^{N} \mathcal{H}_\ell.$$

Such models are characterized by a set of rational numbers:

• The number k appearing in the defining algebra is the level of the conformal field theory and takes values in the positive rational integers $k \in \mathbb{N}$.

• The central charge c in the Virasoro algebra takes rational values $c \in \mathbb{Q}$.

• The anomalous dimensions Δ_ℓ of the physical (chiral primary) fields $\phi_\ell(z,\bar{z})$ on the world-sheet surfaces swept out by the string (torus) are determined to take rational values as well $\Delta_\ell \in \mathbb{Q}$.

The classification of all rational conformal field theories with N=2 worldsheet supersymmetry has not been completed yet. The most notable constructions that have been analyzed so far are based on affine coset constructions, in particular the generalizations of Gepner models introduced by Kazama and Suzuki [22]. In the following the focus will be on the simpler Gepner models.

2.2 SU(2)$_k$ Building Block

The simplest examples of Calabi-Yau manifolds that lead to exactly solvable models are described by extensions of the Virasoro algebra defined by currents of the SU(2) affine theory at some level k, for which the central charge is given by

$$c = \frac{3k}{k+2},$$

and the anomalous dimensions are determined to be

$$\Delta^{\ell}_{(k)} = \frac{\ell(\ell+2)}{4(k+2)}, \qquad \ell = 0, \ldots, k.$$

Associated to the anomalous dimensions are certain conformal field theory characters which are functions on the worldsheet of the string (genus 1 curve). More precisely, consider maps on the upper half-plane $\mathfrak{H}$

$$\chi^\ell : \mathfrak{H} \times \mathbb{C}^2 \longrightarrow \mathbb{C}$$

defined as

$$\chi^\ell(\tau, z, u) := e^{-2\pi i k u} \mathrm{tr}_{\mathcal{H}_\ell} q^{L_0 - \frac{c}{24}} e^{2\pi i u J_0}.$$

These characters can be expressed via the string functions

$$c^\ell_m(\tau) = \frac{1}{\eta^3(\tau)} \sum_{\substack{-|x|<y\leq|x| \\ (x,y) \text{ or } (\frac{1}{2}-x, \frac{1}{2}+y) \\ \in \mathbb{Z}^2 + \left(\frac{\ell+1}{2(k+2)}, \frac{m}{2k}\right)}} \mathrm{sign}(x) e^{2\pi i \tau((k+2)x^2 - ky^2)}$$

and the classical theta functions $\Theta_{m,k}$

$$\Theta_{n,m}(\tau, z, u) = e^{-2\pi imu} \sum_{\ell\in\mathbb{Z}+\frac{n}{2m}} e^{2\pi im\ell^2\tau+2\pi i\ell z}$$

as

$$\chi^\ell(\tau, z, u) = \sum_{\substack{m=-k+1\\ \ell=m \bmod 2}}^{k} c^\ell_m(\tau)\Theta_{m,k}(\tau, z, u). \tag{1}$$

This set of characters is closed under modular transformations. For the translation generator $\tau \mapsto \tau + 1$ the characters acquire a simple scaling factor

$$\chi^\ell(\tau + 1, z, u) = e^{2\pi i\Delta_\ell}\chi^\ell(\tau, z, u)$$

while the S-generator $\tau \mapsto -1/\tau$ leads to the relation

$$\chi^\ell\left(-\frac{1}{\tau}, \frac{z}{\tau}, u + \frac{z^2}{2\tau}\right) = e^{\pi ikz^2/2}\sum_{m=0}^{k} S_{\ell m}\chi^m(\tau, u),$$

with modular S-matrix

$$S_{\ell m} = \sqrt{\frac{2}{k+2}}\ \sin\left(\frac{(\ell+1)(m+1)\pi}{k+2}\right) \tag{2}$$

2.3 N=2 Supersymmetric models

The simplest class of N=2 supersymmetric exactly solvable theories is built in terms of the affine SU(2) theory as a coset model

$$\frac{\mathrm{SU}(2)_k \otimes \mathrm{U}(1)_2}{\mathrm{U}(1)_{k+2,\mathrm{diag}}}. \tag{3}$$

Coset theories G/H lead to central charges of the form $c_G - c_H$, hence the supersymmetric affine theory at level k still has central charge $c_k = 3k/(k + 2)$. The spectrum of anomalous dimensions $\Delta^\ell_{q,s}$ and U(1)−charges Q^ℓ of the primary fields $\Phi^\ell_{q,s}$ at level k is given by

$$\begin{aligned} \Delta^\ell_{q,s} &= \frac{\ell(\ell+2)-q^2}{4(k+2)} + \frac{s^2}{8} \\ Q^\ell_{q,s} &= \frac{q}{k+2} - \frac{s}{2}, \end{aligned} \tag{4}$$

where $\ell \in \{0, 1, \ldots, k\}$, $\ell + q + s \in 2\mathbb{Z}$, and $|q - s| \leq \ell$. Associated to the primary fields are characters defined as

$$\chi^{\ell}_{q,s}(\tau, z, u) = e^{-2\pi i u} \mathrm{tr}_{\mathcal{H}^{\ell}_{q,s}} e^{2\pi i \tau (L_0 - \frac{c}{24})} e^{2\pi i J_0}, \tag{5}$$

where the trace is over the projection of $\mathcal{H}^{\ell}_{q,s}$ to definite fermion number mod 2 of a highest weight representation of the N=2 superfonformal algebra. It is of advantage to express these maps in terms of the string functions and theta functions, leading to the form

$$\chi^{\ell}_{q,s} = \sum c^{\ell}_{q+4j-s} \Theta_{2q+(4j-s)(k+2),2k(k+2)}, \tag{6}$$

because it follows from this representation that the modular behavior of the $N = 2$ characters decomposes into a product of the affine SU(2) structure in the ℓ index and into Θ-function behavior in the charge and sector index. More precisely, the modular behavior of these characters is given by

$$\chi^{\ell}_{q,s}(\tau + 1, z, u) = e^{2\pi i \Delta^{\ell}_{q,s}} \chi^{\ell}_{q,s}(\tau, z, u) \tag{7}$$

for the modular shift $\tau \mapsto \tau + 1$, and by

$$\chi^{\ell}_{q,s}\left(-\frac{1}{\tau}, \frac{z}{\tau}, u + \frac{z^2}{2\tau}\right) = \frac{1}{\sqrt{2}(k+2)} \sum_{\substack{\ell', q', s' \\ \ell'+q'+s'=0 \bmod 2}} e^{\pi i \frac{qq'}{k+2}} e^{-\pi i \frac{ss'}{2}} \sin\left(\pi \frac{(\ell+1)(\ell'+1)}{k+2}\right) \chi^{\ell'}_{q',s'}(\tau, z, u) \tag{8}$$

for the generator $\tau \mapsto -1/\tau$.

Because the transformation behavior of these characters splits into the nontrivial interacting SU(2) theory and a pair of U(1) theories, the partition function of these minimal $N = 2$ models can be written as

$$Z_{N=2}(\tau, \bar{\tau}) = \frac{1}{2} \sum_{\substack{\ell, \bar{\ell} \\ q, \bar{q}, s, \bar{s}}} A_{\ell, \bar{\ell}} M^{q, \bar{q}} M^{s, \bar{s}} \chi^{\ell}_{q,s}(\tau) \chi^{\bar{\ell}}_{\bar{q}, \bar{s}}(\bar{\tau}). \tag{9}$$

2.4 String Compactification with Gepner Models

String theory in a supersymmetric geometric background lives in ten dimensions because of anomaly cancellation constraints. Decompactification of four of the initially ten compact dimensions leaves a compact six-dimensional variety. In the context of theories with $N = 1$ spacetime supersymmetry each of these compact dimensions contributes an anomaly of $3/2$. This means that in order to construct an internal theory based on affine SU(2) theories one has to glue several of these models together in a tensor product structure to provide the correct conformal anomaly. The number of $\mathrm{SU}(2)_{k_i}$-factors one needs to tensor to make a Calabi-Yau variety is determined by the sum

$$c_{\rm tot} = \sum_{i=1}^{r} c_i = \sum_{i=1}^{r} \frac{3k_i}{k_i+2} = 9.$$

The physical spectrum of the massless modes of the compactified theory is in part described by the cohomology of the Calabi-Yau variety. In the conformal field these fields are determined by the two-dimensional chiral primary fields and the dimension of the geometric cohomology theories can be computed by combinatorics of the anomalous dimension of the conformal field theory. More precisely, the multiplicities are determined by the partition function of the theory, which takes the form (in the light cone gauge)

$$Z_{het} =\mid (\mathrm{Im}\ \tau)^{-1}\eta(\tau)^{-4} \mid \sum_{\lambda\in\{o,v,s,\bar{s}\}} B_{\lambda}^{SO(2)}(\tau) B_{\bar{\lambda}}^{E_8\times SO(10)}(\bar{\tau}) Z_{\rm int}(\tau,\bar{\tau}), \qquad (10)$$

where the $4d$–part comes from the spacetime variables. The $SO(2)$ characters $B_{\lambda}^{SO(2)}$ come from the spacetime fermions, whereas the characters $B_{\bar{\lambda}}^{E_8\times SO(10)}$ are determined by the internal fermions, and the internal partition function $Z_{\rm int}$ is determined by the tensor models and will be given by products of the form $\prod_{i=1}^{r} \chi_{q_i,s_i}^{\ell_i}$. Collecting the affine indices into an $\vec{\ell}$–index and the charge and the sector indices (q_i, s_i) as well as the external index (coming from the level 2 theta function of the 4–d fermions) into one $(2r+1)$ vector, we can write the partition function (modulo the external transverse bosons) as

$$\chi^{\vec{\ell}}_{\vec{\mu}} = B^G_{\mu_0} \prod_{i=1}^{r} \chi^{\ell_i}_{\mu_i,\mu_{r+i}}, \tag{11}$$

where G stands for SO(2) in the supersymmetric sector, and for SO(26) or $E_8 \times SO(10)$ in the Kac–Moody sector. The index μ_0 labels the highest weight of a given representation of G. The standard fermionic invariant for the theta–function products then leads to

$$Z \sim \sum_{\vec{\ell},\vec{\bar{\ell}}} A_{\vec{\ell},\vec{\bar{\ell}}} \sum_{\mu,m_i,n_i} (-1)^{m_0+n_0} \chi^{\vec{\ell}}_{\vec{\mu}+m_i\vec{\beta}^i} \chi^{\vec{\bar{\ell}}*}_{\vec{\mu}'+n_i\vec{\beta}^i} \tag{12}$$

for the compactified theory. Consider eq. (12) in more detail. The multi-matrix $A_{\vec{\ell},\vec{\bar{\ell}}} = \prod_{i=1}^{r} A_{\ell_i,\bar{\ell}_i}$ represents a choice of the affine invariant for each of the individual factors in the internal sector. The $(2r+1)$–vectors $\vec{\beta}^i$, $\vec{\mu}$, $\vec{\mu}'$ are chosen in such a way as to implement requirements (B) and (C). To that account define $\vec{\beta}^0 = (\bar{s}|1,\ldots,1)$ and $\vec{\beta}^i = (v|0,\ldots,0,2,0,\ldots,0)$, $\quad i=1,\ldots,r$ where 2 is in the $(r+1+i)^{th}$ entry and $\bar{s}, v$ are highest weights for the group G. The vectors $\vec{\mu}$ are then constrained by

$$\beta^i \cdot \mu \in 2\mathbb{Z} + \delta^{i0} \tag{13}$$

where the scalar product is defined as follows. Recall that the charge indices q_i belong to level $(k+2)$ theta functions whereas the external and the sector indices belong to level 2 theta functions. It is then natural to define

$$\vec{\beta}^i \cdot \mu = \frac{\beta^i_0 \mu_0}{2} + \sum_{j=1}^{r} \left(-\frac{\beta^i_j \mu_j}{k+2} + \frac{\beta^i_{r+j}\mu_{r+j}}{2} \right). \tag{14}$$

For $i=0$ this condition implements the requirement that the U(1) charge must be quantized, while for $i \neq 0$ the requirement of appropriate sector coupling (Neveu-Schwarz to Neveu-Schwarz and Ramond to Ramond) is implemented in the partition function. The integers m_i and n_i run over the integers modulo α where α is such that $\beta_i\alpha/2(k_i+2)$ are integers for all i. The vectors $\vec{\mu}$, $\vec{\mu}'$ only differ in their external entry in such a way as to implement the transition from the superstring to the heterotic string. This is accomplished by permuting the highest weights $(o, v, s, \bar{s})$ of the gauge group G into $(v, o, s, \bar{s})$.

The physical couplings in the conformal field theory are determined mostly (not completely) by the anomalous dimensions of the fields involved.

2.5 An example: the quintic threefold

An example of an exactly solvable Calabi-Yau threefold is the quintic, described as the Fermat type variety in ordinary projective 4-space $\mathbb{P}_4$

$$X_5 = \left\{p(z_i) = z_0^5 + \cdots + z_4^5 = 0\right\} \subset \mathbb{P}_4. \tag{15}$$

In terms of the intermediate Landau-Ginzburg formulation the polynomial $p(z_i)$ originates from the superpotential

$$W(\Phi_i(x,\bar{x})) = \Phi_0^5(x,\bar{x}) + \cdots + \Phi_4^5(x,\bar{x}) \tag{16}$$

in terms of the chiral primary super fields $\Phi_i(x,\bar{x}) = z_i + \psi_i\theta + \cdots$ on the worldsheet parametrized by variables $(x,\bar{x})$. The complex coordinates z_i of the projective variety are obtained by taking the size of the string to zero.

Conformal field theoretically we need theories of level $k = d-2$, where in general d is the degree of the superpotential. In the present case this leads to SU(2) models at level 3, hence a tensor product of five theories is needed $X_5 \cong \mathrm{SU}(2)_3^{\otimes 5}$. The cohomology of the projective variety can be reconstructed in terms of primary fields such that the total anomalous dimension $\Delta_{\mathrm{tot}} = \sum_\ell \Delta_{\ell,0}^\ell = 1$ and the U(1) charges are ± 1.

3 Number Fields from Rational Conformal Field Theories.

The idea in the following is to map part of the conformal field theory data into an interesting number field which can be recovered in a geometric way from the variety.

The starting point is the collection of conformal field theory characters associated to the anomalous dimensions of the primary fields of the theory, more precisely their transformation behavior with respect to the modular transformations on the worldsheet. These transformations involve in particular the generator $S : \tau \mapsto -1/\tau$. The nontrivial behavior of the N=2 characters is carried by their affine SU(2) structure and involves the modular S-matrix (2). (The T-matrix associated to $\tau \mapsto \tau + 1$) does not enter in the present context).

With these S-matrices one can define so-called quantum dimensions

$$Q_\ell = \frac{S_{0\ell}}{S_{00}}$$

and their generalized cousins $Q_{\ell m} = \frac{S_{\ell m}}{S_{0m}}$ for the SU(2) theory at level k.

These numbers are of interest because it is possible to reconstruct the anomalous dimensions from them.

Define the Rogers dilogarithm function as

$$L(z) = Li_2(z) + \frac{1}{2}\log(z)\log(1-z),$$

where

$$Li_2(z) = \sum_{n\in\mathbb{N}} \frac{z^n}{n^2}$$

is Euler's dilogarithm. This function leads to transcendental values when evaluated at algebraic numbers. It is therefore surprising that it can be used to express rational numbers. This however is precisely what has emerged.

Theorem.[Kirillov-Reshetikhin, Lewin]*Let $Q_i = S_{0i}/S_{00}$ be the quantum dimensions of the affine Lie algebra SU(2)$_k$ at level k. Then*

$$\frac{1}{L(1)}\sum_{\ell=1}^{k} L\left(\frac{1}{Q_\ell^2}\right) = \frac{3k}{k+2}. \tag{17}$$

Even more surprising is the result of Kuniba and Nakanishi according to which the anomalous dimensions $\Delta_j^{(k)}$ of the primary fields of the affine SU(2) theory at level k can be reconstructed from the quantum dimensions

$$\frac{1}{L(1)}\sum_{\ell=1}^{k} L\left(\frac{1}{Q_{\ell m}^2}\right) = \frac{3k}{k+2} - 24\Delta_{(k)}^m + 6m. \tag{18}$$

These relations provide an alternative way of parametrizing the anomalous spectrum of fields at fixed central charge via the quantum dimensions. The reason this is of interest in the present context is because the quantum dimensions lead to an interesting field.

Theorem.[Boer & Goeree [11]] *The field extension $K/\mathbb{Q}$ defined by the quantum dimensions of the affine theory $SU(2)_k$ is normal with abelian Galois group* $\mathrm{Gal}(K/\mathbb{Q})$.

Classical results by Kronecker and Weber then immediately determine the type of field that contains the quantum dimensions.

Theorem.[Kronecker-Weber] *The quantum dimension field extension is contained in a cyclotomic extension $\mathbb{Q}(\mu_m)/\mathbb{Q}$ for some integer m.*

This follows from the fusion rules for the chiral primary fields

$$\phi_\ell \star \phi_m = \sum_m N_{\ell m}^r \phi_r,$$

leading to an additive behavior of the anomalous dimensions Δ_ℓ of ϕ_ℓ, and E. Verlinde's result for the fusion matrices [41]

$$N_{\ell m}^n = \sum_m \frac{S_{\ell r} S_{mr}}{S_{0r}} (S^{-1})_{nr}.$$

The alternative parametrization of the anomalous dimensions in terms of the quantum dimensions makes it possible to rephrase the problem of an intrinsic derivation of these characteristics of the conformal field theory as a derivation of the field of quantum dimensions directly from the intrinsic geometry of the variety.

4 The Congruent Zeta Function

The basic idea is to derive the arithmetic structure of the underlying conformal field theory directly from the arithmetic structure of the variety X. The complete diophantine information of X is contained in the cardinalities of the reduced variety $N_{r,p}(X) = \#(X/\mathbb{F}_{p^r})$. These numbers can be used to define a generating function which, in analogy to Dedekind's zeta function of an arbitrary algebraic number field, was introduced in the context for curves over finite fields by E. Artin in 1924 [1] as

$$Z(X/\mathbb{F}_p, t) = \exp\left(\sum_{r\in\mathbb{N}} \#\left(X/\mathbb{F}_{p^r}\right) \frac{t^r}{r}\right). \tag{19}$$

The motivation for arranging the number of solutions of $X/\mathbb{F}_{p^r}$ into the form (19) is that this expression leads to rational functions in the formal variable t. This was first shown for curves C_g of arbitrary genus g by F.K. Schmidt via the Riemann-Roch theorem [35][20]

$$Z(C_g/\mathbb{F}_p, t) = \frac{P^{(p)}(t)}{(1-t)(1-pt)}, \tag{20}$$

where $P^{(p)}(t)$ is a polynomial of degree $2g$.

After much work by Artin, Hasse, Deuring and Weil, culminating in the p-adic analysis of Dwork [14] and Deligne [12], it was shown that more generally the zeta function of d-dimensional smooth projective varieties over finite fields has a number of important structural properties which justify its definition.

1. The zeta function of a variety X of complex dimension d is a rational function in the variable t

$$Z(X/\mathbb{F}_p, t) = \frac{\prod_{j=1}^{d} P_{2j-1}^{(p)}(t)}{\prod_{j=0}^{d} P_{2j}^{(p)}(t)}, \tag{21}$$

where the $P_i^{(p)}(t)$ are polynomials whose degrees are determined by the Betti numbers $b_i = \dim \mathrm{H}^i_{\mathrm{DeRham}}(X)$:

$$\begin{aligned} P_0^{(p)}(t) &= 1-t \\ P_{2d}^{(p)}(t) &= 1-p^d t \end{aligned}$$

and for $1 \leq i \leq 2d-1$

$$P_i^{(p)}(t) = \prod_{j=1}^{b_i} \left(1 - \beta_j^{(i)}(p) t\right),$$

with algebraic integers $\beta_j^{(i)}(p)$.

2. $Z(X/\mathbb{F}_p, t)$ satisfies a functional equation [24]

$$Z\left(X/\mathbb{F}_p, \frac{1}{p^d t}\right) = (-1)^{\chi+\mu} p^{d\chi/2} t^{\chi} Z(X/\mathbb{F}_p, t), \tag{22}$$

where χ is the Euler number of the variety X over the complex numbers $\mathbb{C}$. Furthermore, μ is zero when the dimension d of the variety is odd, and μ is the multiplicity of $-p^{d/2}$

as an eigenvalue of the action induced on the cohomology by the Frobenius automorphism $\Phi : X \longrightarrow X,\ x \mapsto x^p$.

3. The norms of the algebraic integers $\beta_j^{(i)}(p)$ satisfy the Riemann hypothesis [12]

$$|\beta_j^{(i)}(p)| = p^{i/2}, \qquad \forall i. \tag{23}$$

4.1 There are four cases of interest:

(**a.**) Calabi-Yau curves (Elliptic curves)

$$Z(E/\mathbb{F}_p, t) = \frac{P_1^{(p)}(t)}{(1-t)(1-pt)}$$

with quadratic $P_1^{(p)}(t)$. Such curves are used for compactifications of various string theories to eight dimensions and provide a simple framework in which different types of duality conjectures can be investigated in great detail.

(**b.**) Calabi-Yau two-folds with finite fundamental group (K3 surfaces) have a Hodge diamond of the form

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1, \end{array}$$

leading to a congruent zeta function

$$Z(\mathrm{K3}/\mathbb{F}_p, t) = \frac{1}{(1-t)P_2^{(p)}(t)(1-p^2t)}$$

with $P_2^{(p)}$ of degree 22.

(**c.**) Calabi–Yau threefolds with finite fundamental group lead to Hodge diamonds of the form

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & & \\ 1 & & h^{2,1} & & h^{2,1} & & 1, \end{array}$$

resulting in zeta functions of the form

$$Z(X/\mathbb{F}_p, t) = \frac{P_3^{(p)}(t)}{(1-t)P_2^{(p)}(t)P_4^{(p)}(t)(1-p^3t)}$$

with

$$\begin{aligned} deg(P_3^{(p)}(t)) &= 2+2h^{(2,1)} \\ deg(P_2^{(p)}(t)) &= h^{(1,1)} \end{aligned}$$

This follows from the fact that for non-toroidal Calabi-Yau threefolds we have $b_1 = 0$.

(**d.**) Finally, Calabi-Yau fourfolds (with π_1 finite) are of interest because of conjectured dualities involving Vafa's F-theory [39][31]. The general cohomology structure

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & h^{1,1} & & 0 & & \\ & 0 & & h^{2,1} & & h^{2,1} & & 0 & \\ 1 & & h^{3,1} & & h^{2,2} & & h^{3,1} & & 1, \end{array}$$

gives rise to

$$Z(X_4/\mathbb{F}_p, t) = \frac{P_3^{(p)}(t)}{(1-t)P_2^{(p)}(t)P_4^{(p)}(t)P_6^{(p)}(t)(1-p^4t)}$$

with polynomials of the degrees

$$\begin{aligned} \deg P_2^{(p)}(t) &= \deg P_6^{(p)} = h^{(1,1)}, \\ \deg P_3^{(p)}(t) &= 2h^{2,1}, \\ \deg P_4^{(p)}(t) &= 2+2h^{3,1}+h^{2,2}. \end{aligned} \tag{24}$$

4.2 The relation between conformal field theories and Calabi-Yau manifolds in the class of Brieskorn-Pham varieties, or more general polynomials related to Kazama-Suzuki models [28], in general involves singular complete intersections which must be resolved, and/or complete intersections in products of weighted projective spaces. Such varieties have second Betti numbers that can be quite large. The CFT-CY relation however must hold already for the very simplest members in this class in terms of cohomology. All the information necessary to establish such a relation must therefore be present already in these simple examples. Hence we can consider smooth Calabi-Yau hypersurfaces in ordinary projective space

$$\left\{\sum_i z_i^{n+1} = 0\right\} \quad \in \quad \mathbb{P}_{n+1}[n+2]$$

or in smooth hypersurfaces in weighted projective space. Then

$$b_i(X_d) \in \{0,1\}, \quad i < d.$$

For smooth threefolds embedded in weighted projective spaces this leads to a simplified zeta function of the form

$$Z(X/\mathbb{F}_p,t) = \frac{P_3^{(p)}(t)}{(1-t)(1-pt)(1-p^2t)(1-p^3t)} \tag{25}$$

with

$$deg(P_3^{(p)}(t)) = 2 + 2h^{(2,1)} \tag{26}$$

and

$$\sum_{i=1}^{b_3} \beta_i^{(3)} = 1 + p + p^2 + p^3 - \#(X/\mathbb{F}_p). \tag{27}$$

Hence for smooth hypersurface threefolds the only interesting information of the zeta function is encoded in the polynomials $P_3^{(p)}(t)$.

5 Hasse-Weil L-Function

5.1 We see from the rationality of the zeta function that the basic information of this quantity is parametrized by the cohomology of the variety. More precisely, one can show that the i'th polynomial $P_i^{(p)}(t)$ is associated to the action induced by the Frobenius morphism on the i'th cohomology group $\mathrm{H}^i(X)$. In order to gain insight into the arithmetic information encoded in this Frobenius action it is useful to decompose the zeta function. This leads to the concept of a local L-function that is associated to the polynomials $P_i^{(p)}(t)$ via the following definition.

Let $P_i^{(p)}(t)$ be the polynomials determined by the rational congruent zeta function over the field $\mathbb{F}_p$. The i'th L-function of the variety X over $\mathbb{F}_p$ then is defined via

$$L^{(i)}(X/\mathbb{F}_p,s) = \frac{1}{P_i^{(p)}(p^{-s})}. \tag{28}$$

Such L-functions are of interest for a number of reasons. One of these is that often they can be modified by simple factors so that after analytic continuation they (are conjectured to) satisfy some type of functional equation.

5.2 It was mentioned in the previous section that the geometry/CFT relation must hold for the simplest type of varieties, in particular those that do not need any kind of resolution. When considering the cohomology of such simple varieties in dimensions one through four it becomes clear that independent of the dimension of the Calabi-Yau variety the essential ingredient is provided by the cycles that span the middle-dimensional (co)homology. Hence, even though in general the cohomology can be fairly complex, in particular for Calabi-Yau fourfolds, the only local factor in the L-function that is relevant for the present discussion is $P_d^{(p)}(t)$ for $d = \dim_{\mathbb{C}} X$.

This suggests a natural generalization of the concept of the Hasse-Weil L-function of an elliptic curve. Let X be a Calabi-Yau d-fold with $h^{i,0} = 0$ for $0 < i < n-1$ and denote by $\mathfrak{P}(X)$ the set of good prime numbers of X, i.e. those prime numbers for which the variety has good reduction over $\mathbb{F}_p$. Then its associated Hasse-Weil L-function is defined as

$$L_{\mathrm{HW}}(X,s) = \prod_{p\in\mathfrak{P}(X)} \frac{1}{P_d^{(p)}(p^{-s})} = \prod_{p\in\mathfrak{P}(X)} \frac{1}{\prod_{j=1}^{b_3}\left(1-\beta_j^{(d)}(p)p^{-s}\right)}. \tag{29}$$

The existence of bad primes complicates the whole theory considerably, but for our purposes we can ignore the additional factors of the completed L-functions that are induced by these bad primes.

As mentioned above, for smooth weighted CY hypersurfaces the Hasse-Weil L-function then contains the complete arithmetic information of the congruent zeta function. For reasons that will become clear below it is of importance that the above Hasse-Weil function can be related to a Hecke L-function, induced by Hecke characters. The main virtue of such characters is that as the simpler Dirichlet characters they are multiplicative maps. It is this multiplicativity which is of essence for the present framework.

6 The Quintic

The Calabi-Yau variety which we will consider is the quintic hypersurface in ordinary projective fourspace $\mathbb{P}_4$. We denote the general $h^{2,1} = 101$ complex dimensional family of quintic hypersurfaces in projective fourspace $\mathbb{P}_4$ by $\mathbb{P}_4[5]$ and consider the element defined by

$$\mathbb{P}_4[5] \ni X = \left\{ \sum_{i=0}^{4} x_i^5 = 0 \right\}. \tag{30}$$

It follows from Lefshetz's hyperplane theorem that the cohomology below the middle dimension is inherited from the ambient space. Thus we have $h^{1,0} = 0 = h^{0,1}$ and $h^{1,1} = 1$, while $h^{2,1} = 101$ follows from counting monomials of degree five. Following Weil [42] the zeta function is determined by (25), where the numerator is given by the polynomial $P_3^{(p)}(t) = \prod_{i=1}^{204}(1 - \beta_i^{(3)}(p)t)$ which takes the form

$$P_3^{(p)}(t) = \prod_{\alpha \in \mathcal{A}} (1 - j_p(\alpha)t)\,. \tag{31}$$

This expression involves the following ingredients. Define $\ell = (5, p-1)$ and rational numbers α_i via $\ell\alpha_i \equiv 0(\text{mod } 1)$. The set $\mathcal{A}$ is defined as

$$\mathcal{A} = \{\alpha = (\alpha_0, ..., \alpha_4) \mid 0 < \alpha_i < 1,\ \ell\alpha_i \equiv 0(\text{mod } 1), \sum_i \alpha_i = 0(\text{mod } 1)\}. \tag{32}$$

Defining the characters $\chi_{\alpha_i} \in \hat{\mathbb{F}}_p$ in the dual of $\mathbb{F}_p$ as $\chi_{\alpha_i}(u_i) = exp(2\pi i\alpha_i s_i)$ with $u_i = g^{s_i}$ for a generating element $g \in \mathbb{F}_p$, the factor $j_p(\alpha)$ finally is determined as

$$j_p(\alpha) = \frac{1}{p-1} \sum_{\sum_i u_i = 0} \prod_{i=0}^{4} \chi_{\alpha_i}(u_i). \tag{33}$$

We thus see that the congruent zeta function leads to the Hasse-Weil L-function associated to a Calabi-Yau threefold

$$L_{\text{HW}}(X,s) = \prod_{p \in \mathfrak{P}(X)} \prod_{\alpha \in \mathcal{A}} \left(1 - \frac{j_p(\alpha)}{p^s}\right)^{-1}, \tag{34}$$

ignoring the bad primes, which are irrelevant for our purposes.

7 Algebraic Hecke Characters

As mentioned in the introduction, the for the uninitiated surprising aspect of the Hasse-Weil L-function is that it is determined by another, a priori completely different, kind of L-function.

This second type of L-function is derived not from a variety but from a number field. It is this possibility to interpret the cohomological Hasse-Weil L-function as a field theoretic L-function which establishes the connection that allows to derive number fields K from algebraic varieties X.

For the case at hand the type of L-function that is relevant is that of a Hecke L-function determined by a Hecke character, more precisely an algebraic Hecke character. Following Weil we will see that the relevant field for our case is the extension $\mathbb{Q}(\mu_m)$ of the rational integers $\mathbb{Q}$ by roots of unity, generated by $\xi = e^{2\pi i/m}$ for some rational integer m. It turns out that these fields fit in very nicely with the conformal field theory point of view. In order to see how this works this Section first describes the concept of Hecke characters and then explains how the L-function fits into this framework.

There are many different definitions of algebraic Hecke characters, depending on the precise number theoretic framework. Originally this concept was introduced by Hecke [21] as Grössencharaktere of an arbitrary algebraic number field. In the following Deligne's adaptation of Weil's Grössencharaktere of type A_0 is used [13].

Definition. *Let $\mathcal{O}_K \subset K$ be the ring of integers of the number field K and $\mathfrak{f} \subset \mathcal{O}_K$ an ideal. Denote by $\mathfrak{I}(K)$ the set of ideals of K. An algebraic Hecke character modulo $\mathfrak{f}$ is a multiplicative function χ defined on the ideals $\mathfrak{I}(K)$ that are relatively prime to $\mathfrak{f}$ for which the following condition holds. There is an element $\sum n_\sigma \sigma \in \mathbb{Z}[\mathrm{Gal}(K/\mathbb{Q})]$ in the integral group ring of the Galois group of the abelian extension $K/\mathbb{Q}$ such that if $\alpha \in \mathcal{O}_K$, $\alpha \equiv 1 (\mathrm{mod}\ \mathfrak{f})$ then*

$$\chi((\alpha)) = \prod_\sigma \sigma(\alpha)^{n_\sigma}. \tag{35}$$

Furthermore, there is an integer $w > 0$ such that $n_\sigma + n_{\bar{\sigma}} = w$ for all $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$. This integer w is called the weight of the character χ.

Given any such character χ defined on the ideals of the algebraic number field K we can follow

Hecke and consider a generalization of the Dirichlet series via the L-function

$$L(\chi, s) = \prod_{\substack{\mathfrak{p} \subset \mathcal{O}_K \\ \mathfrak{p} \text{ prime}}} \frac{1}{1 - \frac{\chi(\mathfrak{p})}{\mathrm{N}\mathfrak{p}^s}} = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{\chi(\mathfrak{a})}{\mathrm{N}\mathfrak{a}^s}, \tag{36}$$

where the sum runs through all the ideals. Here $\mathrm{N}\mathfrak{p}$ denotes the norm of the ideal $\mathfrak{p}$, which is defined as the number of elements in $\mathcal{O}_K/\mathfrak{p}$. The norm is a multiplicative function, hence can be extended to all ideals via the prime ideal decomposition of a general ideal. If we can deduce from the Hasse-Weil L-function the particular Hecke character(s) involved we will be able to derive from the varieties distinguished number field K.

Insight into the nature of number fields can be gained by recognizing that for certain extensions K of the rational number $\mathbb{Q}$ the higher Legendre symbols provide the characters that enter the discussion above. Inspection then suggests that we consider the power residue symbols of cyclotomic fields $K = \mathbb{Q}(\mu_m)$ with integer ring $\mathcal{O}_K = \mathbb{Z}[\mu_m]$. The transition from the cyclotomic field to finite fields is provided by the character which is determined for any algebraic integer $x \in \mathbb{Z}[\mu_m]$ prime to m by the map

$$\chi_\bullet(x) : \mathfrak{I}_m(\mathcal{O}_K) \longrightarrow \mathbb{C}^*, \tag{37}$$

which is defined on ideals $\mathfrak{p}$ prime to m by sending the prime ideal to the m'th root of unity for which

$$\mathfrak{p} \quad \mapsto \quad \chi_\mathfrak{p}(x) = x^{\frac{\mathrm{N}\mathfrak{p}-1}{m}} (\mathrm{mod}\ \mathfrak{p}). \tag{38}$$

Using these characters one can define Jacobi-sums of rank r for any fixed element $a = (a_1, ..., a_r)$ by setting

$$J_a^{(r)}(\mathfrak{p}) = (-1)^{r+1} \sum_{\substack{u_i \in \mathcal{O}_K/\mathfrak{p} \\ \sum_i u_i = -1 (\mathrm{mod}\ \mathfrak{p})}} \chi_\mathfrak{p}(u_1)^{a_1} \cdots \chi_\mathfrak{p}(u_r)^{a_r} \tag{39}$$

for prime $\mathfrak{p}$. For non-prime ideals $\mathfrak{a} \subset \mathcal{O}_K$ the sum is generalized via prime decomposition $\mathfrak{a} = \prod_i \mathfrak{p}_i$ and multiplicativity $J_a(\mathfrak{a}) = \prod_i J_a(\mathfrak{p}_i)$. Hence we can interpret these Jacobi sums as a map $J^{(r)}$ of rank r

$$J^{(r)} : \mathfrak{I}_m(\mathbb{Z}[\mu_m]) \times (\mathbb{Z}/m\mathbb{Z})^r \longrightarrow \mathbb{C}^*, \tag{40}$$

where $\mathfrak{I}_m$ denotes the ideals prime to m. For fixed $\mathfrak{p}$ such Jacobi sums define characters on the group $(\mathbb{Z}/m\mathbb{Z})^r$. It can be shown that for fixed $a \in (\mathbb{Z}/m\mathbb{Z})^r$ the Jacobi sum $J_a^{(r)}$ evaluated at principal ideals (x) for $x \equiv 1(\mathrm{mod}\ m^r)$ is of the form $x^{S(a)}$, where

$$S(a) = \sum_{\substack{(\ell,m)=1 \\ \ell \bmod m}} \left[\sum_{i=1}^{r} \left\langle \frac{\ell a_i}{m} \right\rangle\right] \sigma_\ell^{-1}, \tag{41}$$

where $< x >$ denotes the fractional part of x and $[x]$ describes the integer part of x.

We therefore see that the Hasse-Weil L-function is in fact a product of functions each of which is determined by a Hecke character defined by a Jacobi sum that is determined by a prime ideal in the cyclotomic field $\mathbb{Q}(\mu_m)$. In the case of the quintic hypersurface we derive in this way the fusion field from the arithmetic structure of the defining variety reduced to a finite field. To summarize, we have seen that the fusion field of the underlying conformal field theory is precisely that number field which is determined when the cohomological Hasse-Weil L-function is interpreted as the Hecke L-function associated to an algebraic number field.

8 Quantum Dimensions

8.1 The discussion so far leads to the question whether there is a natural field theoretic interpretation of the quantum dimensions. The quantum dimensions are real, hence one needs to consider the real field

$$\mathbb{Q}(\mu_m)^+ \subset \mathbb{Q}(\mu_m),$$

generated by $(\xi_m + \xi_m^{-1})$.

This field naturally emerges via the class number of the cyclototomic field, which splits as

$$h(\mathbb{Q}(\mu_m)) = h^+\ h^-$$

where

$$h^+ = h(\mathbb{Q}(\mu_m))^+.$$

The class number h^+ in turn singles out within the real field $\mathbb{Q}(\mu_m)^+$ the cyclotomic units

$$\theta_j = \left|\frac{1-\xi_m^j}{1-\xi_m}\right| = \frac{\sin\frac{j\pi}{m}}{\sin\frac{\pi}{m}}.$$

These numbers are precisely the quantum dimensions!

The exploration of this has a long history, more than 150 years, starting with Kummer.

Theorem.[Kummer 1847, Sinnott 1978] *Let $m > 2$, $m \neq 2 \pmod 4$ be the conductor of $\mathbb{Q}(\mu_m)$. Denote by U_c^+ the subgroup spanned by the cyclotomic units within the group U^+ of positive real units in $\mathbb{Q}(\mu_m)$. Then*

$$h^+ = 2^b[U^+ : U_c^+],$$

where $b = 0$ if the number g of prime factors is unity, and

$$b = 2^{g-2} + 1 - g$$

if $g > 1$.

This provides an identification of the quantum dimensions.

8.2 Furthermore, the class number h^+ of the maximal real subfield $\mathbb{Q}(\mu_p)^+$ is constructed in part from the real cyclotomic units. For p an odd prime

$$h^+ = \frac{2^{(p-3)/2}\Delta}{R},$$

where R is the regulator of the field, in general defined as the logarithmic image of a set of fundamental units and Δ is a determinant constructed from the quantum dimensions

$$\Delta \;=\; \left|\det\left(\sigma_j(\theta_k)\right)_{\substack{2\leq k\leq (p-1)/2\\ 0\leq j\leq (p-3)/2}}\right|. \tag{42}$$

The regulator R can be viewed as the volume of the logarithmic image of a fundamental system of units. It was shown by Dirichlet that the group U of units in an algebraic number field of degree $[K:\mathbb{Q}] = r_1 + 2r_2$ takes the form

$$U \cong \mu \prod_{i=1}^{r_1+r_2-1} G_i, \tag{43}$$

where μ is the group of roots of unity, each G_i is a group of infinite order, and r_1 (r_2) denotes the number of real (complex) embeddings of the field K. Hence every unit $u \in U$ can be written in the form $u = \alpha \prod_{i=1}^r \epsilon_i$, where $\alpha \in \mu$ and $\{\epsilon_i\}_{i=1,\ldots,r=r_1+r_2-1}$ is called a fundamental system of units. It is useful to translate the multiplicative structure of the units into an additive framework via the regulator map

$$r : U \longrightarrow \mathbb{R}^{r_1+r_2} \tag{44}$$

defined by

$$r(u) = \left(\ln|\rho_1(u)|, \ldots, \ln|\rho_{r_1}(u)|, \ln|\rho_{r_1+1}(u)|^2, \ldots, \ln|\rho_{r_1+r_2}(u)|^2\right), \tag{45}$$

where $\{\rho_i\}_{i=1,\ldots,r_1}$ are the real embeddings and $\{\rho_{r_1+j}\}_{j=1,\ldots,r_2}$ are the complex embeddings of K. The regulator R then is defined as

$$R = \det\left(a_i \ln|\rho_i(\epsilon_j)|\right)_{\substack{1\leq i\leq r_1+r_2 \\ 1\leq j\leq r_1+r_2-1}}, \tag{46}$$

with $a_i = 1$ for the real embeddings $i = 1, ..., r_1$, and $a_i = 2$ for the complex embeddings $i = r_1+1, ..., r_1+r_2$. The regulator is independent of the choice of the fundamental system of units.

8.3 The results described above provide an example in which the class number of an algebraic number field acquires physical significance. This is not without precedence. Recently the class number of the fields of definition of certain arithmetic black hole attractor varieties have been interpreted as the number of U-duality classes of black holes with the same area [30]. Here we see a further instance where the class number of an algebraic number field acquires physical meaning.

9 Beilinson-Bloch Conjectures

9.1 The idea of analyzing the conformal field theory structure of Calabi-Yau varieties in terms of their arithmetic properties originally suggested itself [33] via some work by Bloch and Schoen that was aimed at confirming some of the conjectures of Beilinson [2] and Bloch [3]. It was shown in refs. [4] and [36] in the context of the resolution of a nodal quintic threefold that the vanishing order of the Hasse-Weil function at the central value determines the rank of a twisted cohomology group, where the twist is determined by a quadratic extension of the rational numbers.

More precisely, one has the following

Conjecture. (Swinnerton–Dyer, Beilinson, Bloch)
Consider a complex variety X and denote by $\mathrm{CH}^r(X)$ the Chow group of codimension r algebraic cycles modulo rational equivalence. Let furthermore $\mathrm{CH}^r(X)_0$ denote the subgroup of nullhomologous cycles and $\mathrm{L}(\mathrm{H}^{2r-1}, s)$ be the L−function associated to the cohomology group H^{2r-1}. Then

$$\mathrm{rk}\ \mathrm{CH}^r(X)_0 = \mathrm{ord}^{\mathrm{zero}}_{s=r}\ \mathrm{L}(\mathrm{H}^{2r-1}, s).$$

More precisely, these conjectures equate the vanishing order of the L–function associated to an odd dimensional cohomology group $\mathrm{H}^{2r-1}(X_{\bar{K}}, \mathbb{Q}_\ell)$ of a smooth projective variety X over a number field K at the point $s = r$ to the rank of the group

$$A^r(X) = Ker\left(\mathrm{CH}^r(X_K) \otimes \mathbb{Q} \longrightarrow H^{2r}(X_{\bar{K}}, \mathbb{Q}_\ell(r))\right),$$

where $\mathrm{CH}^r(X)$ is the Chow group of codimension r cycles on X defined over K modulo rational equivalence. Then

$$\mathrm{rk}\ \mathrm{A}(X) = \mathrm{ord}_{s=r}\ \mathrm{L}(\mathrm{H}^{2r-1}, s).$$

9.2 Example. A tantalizing check for the Beilinson-Bloch conjectures was considered by Bloch [4] and Schoen [36]. Their starting point is the deformed quintic at the conifold point

$$Y_0 = \{z_0^5 + \cdots + z_4^5 - 5\prod_{i=0}^{4} z_i = 0\}\ \subset \mathbb{P}_4. \qquad (47)$$

This variety fails to transverse at 125 nodal points

$$\left\{(\xi_0, ..., \xi_4) \mid \xi_i^5 = 1,\ \prod_{i=1}^{4} = 1\right\}.$$

The resolution was described in detail by Schoen [36] and was found to lead to a rigid manifold Y with Euler number $\chi(Y) = 50$ and Hodge diamond

$$
Y: \quad \begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 25 & & 0 & \\
1 & & 0 & & 0 & & 1 \\
 & 0 & & 25 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
$$

The interesting thing now is that even though this variety is rigid over $\mathbb{C}$, the intermediate cohomology does not vanish when one considers cohomology with weird fields. Namely, it was shown by Bloch and Schoen that the following holds.

Theorem.[Bloch]
1. $L(H^3(Y), 2) \neq 0$.
2. *If ρ is the quadratic character associated to $\mathbb{Q}(\sqrt{5})/\mathbb{Q}$ then $L(\rho \otimes H^3(Y), s)$ vanishes to order 1 at $s = 2$.*

It follows from standard conjectures and the theorem that the Griffiths group in codimension 2 is of rank 0 over $\mathbb{Q}$ but has rank 1 over $\mathbb{Q}(\sqrt{5})$. Assuming the purity conjecture for étale cohomology the cycle implied by this result has infinite order in the Griffiths group. In this way a nonzero map is obtained

$$A^2(X_{\mathbb{Q}(\sqrt{5})}) \longrightarrow \mathbb{Z}/3\mathbb{Z}.$$

The important point of all this is that these arithmetic considerations single out the quadratic field $\mathbb{Q}(\sqrt{5})$, which is precisely the field generated by quantum dimensions of the Gepner model.

This is the good news. The bad news is that the significance of this result is not obvious.

(1) The starting point is a deformed quintic, not the Fermat quintic. Only the latter is known to be exactly solvable. Deformed conformal field theories are still poorly understood, hence one could question the significance of this result. One argument in favor of its relevance is that the quantum dimensions are indicative of cohomology, hence deformations per se should not produce dramatic behavior (unless one hits a singular point).

(2) Worse, the starting point is nodal, precisely when drastic things do happen, and the computation is done for the resolved variety.

In light of these difficulties the obvious question arises whether there are other varieties which feature a similar behavior and which can be related to exactly solvable field theories. In this context a conjecture by Yui is rather interesting.

Conjecture.[Yui]*For any Brieskorn-Pham threefold defined over* $\mathbb{Q}$ *there exists a character* ρ *associated to the cyclotomic field such that*

$$\mathrm{ord}_{s=2}\ \mathrm{L}(\rho \otimes \mathrm{H}^3(X_{\mathbb{Q}}), 2) \geq 1.$$

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