Optimal recovery of solutions to Dirichlet Problems for Laplace's Equation

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Laplace's equation

- The following equation is called Laplace's equation

$$\Delta U = 0.$$

Which is a partial differential equation of second order.

- The Laplacian of U is $\Delta U = \sum_{i=1}^{n} U_{x_i x_i}$
- The solution U is a C^2 harmonic function.

$$U : \bar{\Omega} \to \mathbb{R}$$

$$X \mapsto U(x)$$

Where Ω is a given open set, $\bar{\Omega} \subset \mathbb{R}^n$

Applications of Laplace's Equation

- Steady-State Heat Conduction.
- Statics Deflection of a Membrane.
- Electrostatic Potential.

Dirichelt's problem

When we add the boundary condition to Laplace's equation, we obtain Dirichlet's problem:

$$\Delta U = 0 \text{ in } \Omega$$

$$U = f \text{ on } \partial\Omega$$

The solution to this problem is well known. For every element $\mathbf{x}$ of the domain we have

$$U(\mathbf{x}) = -\int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{y}$$

Where f is the given function, $\frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \bar{n}}$ is the normal derivative of the Green function.

Questions

Question: what if the function f is not fully known?

- What if f is known at n fixed points?
- What if f is known at n points, but with error ?
- What if f is known at n points, but we have a choice where to take those measurements without error?
- What if f is known at n first Fourier coefficients?

This project: we assumed that f is known at n points on the boundary $\partial\Omega$:

$$f(x_1), f(x_2), ..., f(x_n) \text{ and } f \in H^{\omega}(\partial\Omega)$$

Where H^{ω} is a certain class of smoothness defined with the help of ω.

Goal: develop an optimal method of recovery of solution to the Dirichlet problem based on incomplete information about f and to compute the (optimal) error.

Definitions

Definition

Method of recovery is an operator such that

$$\Phi : \mathbb{R}^n \to C^2(\Omega) \cap H^\omega(\partial\Omega), \qquad \bar{y} \mapsto \Phi(\bar{y})$$

Definition

Error of recovery by method Φ *based on the given information* $\bar{y} = (f(x_1), f(x_2), ..., f(x_n))$ *is defined in* $\partial\Omega$.

$$E(\Phi, \bar{y}) = \sup_{f \in H^\omega} \|U(\mathbf{x}) - \Phi(\bar{y})\|$$

Where Φ *is the method of recovery,* $\bar{y}$ *is the given data point, and* U *is the actual solution.*

Definition

The optimal error

$$E(\bar{y}) = \inf_{\Phi} E(\Phi, \bar{y})$$

Main results

1. Construct an Extremal Function

$$f_\rho(\mathbf{x}) = min\{\omega(|\mathbf{x} - x_i|)\}, \qquad 1 \leqslant i \leqslant n$$

In every cell

$$\Pi_i = \{x : f_\rho(\mathbf{x}) = \omega(\rho(\mathbf{x}, x_i)\}, \quad 1 \leqslant i \leqslant n$$

$$C_f(\mathbf{y}) = y_i \ \text{ On } \ \Pi_i$$

2. Method of Recovery

$$\Phi^\star : \tilde{U}(\mathbf{x}) \simeq -\int_{\partial\Omega} C_f(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} dy$$

Where C_f is an approximant to f

Theorem

The optimal error

$$\inf_{\Phi} \sup_{f \in H^\omega} \|U(\mathbf{x}) - \Phi(\bar{y})\|_{L_1} = \|U(\mathbf{x}) - \tilde{U}(\mathbf{x})\|_{L_1} = \int_{\partial\Omega} f_\rho(\mathbf{y}) (\frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{x}) d\mathbf{y}$$

Proof: Estimate from Above Part 1

$$\|U(\mathbf{x}) - \tilde{U}(\mathbf{x})\|_{L_1} = \int_\Omega \left| U(\mathbf{x}) - \tilde{U}(\mathbf{x}) \right| dx$$

$$= \int_\Omega \left| - \int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{y} + \int_{\partial\Omega} C_f(\mathbf{y}) \frac{\partial G \mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{y} \right| d\mathbf{x}$$

$$= \int_\Omega \left| \int_{\partial\Omega} (C_f(\mathbf{y}) - f(\mathbf{y})) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{y} \right| d\mathbf{x}$$

We have

$$\int_{\partial\Omega} (C_f(\mathbf{y}) - f(\mathbf{y})) d\mathbf{y} = \sum_{i=1}^{n} \int_{\Pi_i} (y_i - f(\mathbf{y})) d\mathbf{y}$$

Estimate on each $\Pi_i, 1 \leqslant i \leqslant n$

$$|f(\mathbf{y}) - y_i| = |f(\mathbf{y}) - f(x_i)|$$

On every cell Π_i, we have

$$|f(\mathbf{y}) - f(x_i)| \leqslant \omega(\rho(\mathbf{y}, x_i)) = f_\rho(\mathbf{y})$$

Proof: Estimate from Above Part2

Therefore on every cell we have $\Pi_i : |f(\mathbf{y}) - y_i| \leqslant f_\rho(\mathbf{y})$

$$\|U(\mathbf{x}) - \tilde{U}(\mathbf{x})\|_{L_1} \leqslant \int_\Omega \left| \int_{\partial\Omega} f_\rho(\mathbf{y}) \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{y} \right| d\mathbf{x}$$

$$\leqslant \int_\Omega f_\rho(\mathbf{y}) \left(\int_{\partial\Omega} \frac{\partial G(\mathbf{x}, \mathbf{y})}{\partial \bar{n}} d\mathbf{x} \right) d\mathbf{y}$$

Proof: Estimate from Below

Estimate from below

$\sup_{f\in H^\omega} \| -\int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial(\bar{n})} d\mathbf{y} - \Phi(\bar{\mathbf{y}})\| \geq \sup_{f\in H^\omega} \| -\int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial(\bar{n})} d\mathbf{y} - \Phi(0)\|$

$$= \sup_{f\in H^\omega} max(\| -\int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \bar{n}} d\mathbf{y} - \Phi(0)\|, \| \int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \bar{n}} d\mathbf{y} - \Phi(0)\|)$$

Using the fact that $\max(\|a\|, \|b\|) \geq \frac{1}{2}(\|a\| + \|b\|)$

$$\geq \sup_{f\in H^\omega} \frac{1}{2}(\| -\int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \bar{n}} d\mathbf{y} - \Phi(0)\| + \| \int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \bar{n}} d\mathbf{y} - \Phi(0)\|)$$

Using the fact that $(\|a\| + \|b\|) \geq \|a - b\|$

$$\geq \sup_{f\in H^\omega} \frac{1}{2}\| -2\int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \bar{n}} d\mathbf{y} - \Phi(0) + \Phi(0)\|$$

$$\geq \sup_{f\in H^\omega} \| \int_{\partial\Omega} f(\mathbf{y}) \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \bar{n}} d\mathbf{y}\|$$

$$\geq \| \int_{\partial\Omega} f_\rho(\mathbf{y}) \frac{\partial G(\mathbf{x},\mathbf{y})}{\partial \bar{n}} d\mathbf{y}\|$$